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Life–prediction for constant–stress fatigue in carbon–fibre composites

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Life-prediction for constant-stress fatigue in carbon-fibre composites

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A detailed analysis has been carried out of the strength and fatigue behaviour under constant-stress cycling of four modern aerospace CFRP laminates, all having the common lay-up $[(\pm 45, 0_2)_2]_S$. Despite some important differences in their basic material characteristics, the fatigue responses of the four materials were similar. This pattern of behaviour has resulted in the development of a descriptive model of constant-life fatigue which has then been used as the basis for a life-prediction procedure. Preliminary attempts at validation of the method have met with a reasonable degree of success and suggest that it could provide designers with a means of selecting newly developed composites for fatigue applications on the basis of far less experimental data than are currently needed for confidence in design.

1. Introduction

Increased exploitation of fibre composites in aerospace and automotive engineering offers significant benefits, among which we may include improved stiffness-to-weight ratio, strength-to-weight ratio and toughness or impact resistance as the most important among the group of mechanical properties and, in appropriate cases, reduced overall cost as the main commercially important factor. In combination, some or all of these benefits provide strong driving forces for a manufacturing company to

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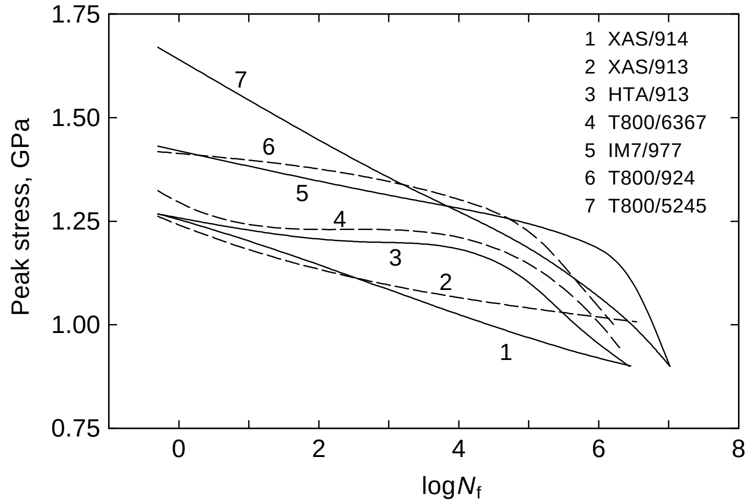

Figure 1. Stress–median-life curves at a stress ratio $R = +0.1$ for seven varieties of carbon-fibre composite of lay-up $[(\pm 45, 0_2)_2]_S$.

re-evaluate its designs and its production processes. For applications requiring long-term stability, whether it be thermal, mechanical or chemical, the decision to make a switch from conventional metallic systems to polymer-based composites may cause great difficulty because the information on which to base new designs may be either partly or totally lacking.

Fatigue is of particular concern to designers working on load-bearing applications where variable stresses are present. Also, while it is true that fatigue studies have been carried out on composites since these materials first began to be studied as serious engineering materials, it is still by no means possible to make safe predictions for materials which have not previously been the subject of extensive investigation. An example will illustrate this. There is a familiar postulate, referred to as the strength-life equal-rank assumption, which supposes that if a series of composite materials are ranked according to their tensile strengths, then their fatigue responses will be similarly ranked. The danger of this can be seen from the group of stress–median-life ($\sigma/\log N_f$) curves for a range of carbon-fibre composites consisting of various combinations of fibres and resins, but of similar lay-ups (namely, $[(\pm 45, 0_2)_2]_S$) shown in figure 1. It can be seen that the $\sigma/\log N_f$ curves are of different shapes; that there is no obvious pattern of responses that can be linked to specific fibre types and that the resin matrix exerts stronger effects, in modifying the apparent performance of a given variety of fibre, than might have been expected. The individual curves also sometimes cross and recross, which means that the strength/life assumption cannot be valid. This is emphasized by looking specifically at fatigue stresses for given lives or, equivalently, fatigue lives for selected cyclic stresses. While there is a general trend that higher strength laminates have longer fatigue lives, in detail it would be unsafe to rely on such a crude ranking procedure.

Another defect in the ranking model is that it attempts to relate only tensile strength to fatigue life; it gives no clues as to how a given material might behave under compression or combined tension–compression fatigue. It is known that the compression strength of a composite laminate will usually be much lower than its

Table 1. *Ratio of compression strength to tensile strength, σ_c/σ_t, for some CFRP laminates of* $[(\pm 45, 0_2)_2]_S$ *construction*

material	σ_c/σ_t
HTA/913	0.77
T800/924	0.63
IM7/977	0.64
T800/5254	0.53

tensile strength, but it is by no means clear that there is a simple relationship between the two, as illustrated in table 1 for some of the composites represented in figure 1.

Fatigue experiments are expensive of time and resources. When a newly developed composite material is being considered for a given application, fatigue data are ideally required at an early stage of the design process and yet a detailed fatigue profile is seldom available until late in the development of a project. There has been a great deal of research on methods of life prediction for composites and recent reviews of various models can be found, for example, in Reifsnider (1991). But few existing models are readily applicable to a wide range of materials and realistic applications and components. Any generally useful model should have the capability for conservative prediction, with appropriate statistical safeguards, from as small an experimental data base as possible, and with the ability to upgrade its predictions smoothly as new fatigue data become available. In this paper, we describe ideas relating to a model that we have developed over a number of years and which appears to be applicable to a variety of kinds of composite laminate.

In a recent paper (Gathercole *et al.* 1994), we examined the constant-stress-amplitude fatigue response of a T800/5245 composite laminate consisting of a high-failure-strain carbon fibre in a toughened bismaleimide resin with a $[(\pm 45, 0_2)_2]_S$ lay-up. Stress–life curves were presented for a range of stress ratios ($R = \sigma_{\min}/\sigma_{\max}$) from repeated tension to repeated compression and, from these data, parametric constant-life curves were derived which provided an empirical relationship between the alternating and mean components of stress which can be used for design purposes. A limited comparison with results for some early laminate materials, XAS-carbon/epoxy, Kevlar-49/epoxy and hybrids of the two, in unidirectional lay-up, suggested that the procedure may have some general level of validity (Adam *et al.* 1986, 1989; Fernando *et al.* 1988). In continuing this work, we have examined a wider range of composites with a view to assessing the potential usefulness of the technique and the level of confidence that a designer using it might have in making preliminary predictions of life from limited data sets. In this paper, further results are presented for three other modern carbon-fibre-reinforced plastics (CFRP), all of the same $[(\pm 45, 0_2)_2]_S$ lay-up as the T800/5245 laminate discussed by Gathercole *et al.* (1994). The parameters of the constant-life relationship for all five of these laminates are compared and an illustration is given of how the method may be used for life prediction.

This paper does not attempt to address the problems of understanding the microstructural mechanisms of fatigue failure in composite materials. For physical insight leading to the development of improved materials, it is clear that this will always be necessary. From the point of view of the designer, however, it is not necessary.

Table 2. *Experimental materials and their sources*

(Note: 924 resin is similar to 6376 (also produced by Ciba), to which occasional reference is made in this paper.)

material	fibre	matrix	supplier
HTA/913	ENKA high strength, standard modulus Tenax fibre	BSL 913 standard epoxy (low-T cure)	Ciba–Geigy (UK)
T800/5245	Toray intermediate modulus, high failure strain	Narmco epoxy/bis-maleimide	BASF (Europe)
T800/924	Toray intermediate modulus, high failure strain	Ciba 924 high-strain, toughened epoxy (high-T cure)	Ciba–Geigy (UK)
IM7/977	Hercules intermediate modulus, high failure strain	977 modified epoxy	Fiberite (Europe)

2. Materials and testing procedures

(*a*) *Experimental materials*

The three laminates for which new results are discussed in this paper are the carbon-fibre composites HTA/913, T800/924 and IM7/977. Details of these materials, together with relevant information for the T800/5245 studied by Gathercole *et al.* (1994), are given in table 2. Within this group there is one representative of the older-established variety of composite, HTA/913, which is similar in many respects to the T300/913 that has been the subject of many research programmes. The higher-performance fibres, T800 and IM7, have similar character but are embedded in quite different resins. Among these four laminates, therefore, we have the opportunity to investigate the specific effects of fibre and resin characteristics on the fatigue performance of the composite and on the applicability of the life-prediction model which we are presenting. Brief mention will also be made of results obtained for a second material reinforced with HTA fibres, in this case in an alternative 982 resin, used in some cases as a substitute for HTA/913.

The composite prepregs were all notionally zero-bleed materials and the final 16-ply $[(\pm 45, 0_2)_2]_S$ laminates, autoclaved according to the suppliers' recommendations, were of nominal fibre volume fractions, V_f, between 0.65 and 0.69 and of thickness approximately 3 mm. Test pieces for strength and fatigue tests were cut to nominal dimensions of 200 mm × 20 mm with a water-cooled diamond saw. Regions near to plate edges and defective areas identified by C-scanning were excluded from the test programme. The cut edges of samples were lightly polished and, after abrasion of the end surfaces, end tabs of 1.6 mm thick soft aluminium were glued on with Ciba–Geigy Redux 403 epoxy-resin paste. The adhesive was cured in a dry oven at 40 °C for 24 h. The central gauge sections of the test samples were 100 mm long.

(*b*) *Testing procedures*

Fatigue tests were carried out in Instron 1300 series servo-hydraulic machines capable of constant load (±100 kN) or constant deflexion (±50 mm) cycling. Testing was carried out at frequencies between about 2 and 10 Hz. A programmable signal generator and analyser (SArGen, by Marandy Instruments, Bath, UK) directs a volt-

Table 3. *Mechanical properties of experimental CFRP laminates*
(Note: standard deviations in brackets.)

	material			
property	HTA/913	T800/5254	T800/924	IM7/977
tensile strength, GPa	1.27 (0.05)	1.67 (0.09)	1.42 (0.09)	1.43 (0.07)
tensile modulus, GPa	69.8 (4.4)	94.0 (3.1)	92.0 (8.0)	90.2 (11.3)
failure strain, %	1.7 (0.1)	1.7 (0.1)	1.5 (0.1)	1.5 (0.1)
compression strength, GPa	0.97 (0.08)	0.88 (0.10)	0.90 (0.090)	0.90 (0.07)

age to the fatigue machine actuator (±10 V maximum range, tension-compression), by-passing the machine's internal signal generator. Outputs from load cell or strain sensors are returned to the generator, recorded, and compared with preset levels, the output control signal to the actuator then being adjusted, as necessary, to ensure that the two remain coincident. A Macintosh computer acts as a user interface which supervises SArGen and stores data on hard disc.

In constant-amplitude fatigue testing, the required frequency and maximum and minimum levels of the load cycle are inputs to the Macintosh. A D/A converter in the SArGen converts this information into a sinusoidal voltage signal which is directed to the Instron actuator. The load-cell output signal (and up to three additional output signals) are directed to an A/D converter in the SArGen and these signals are converted to 180 data points per cycle. The load maxima and minima are recorded and if the averages of five consecutive maxima and minima are different from preset values (for example, as a result of changing specimen compliance) the output signal to the actuator is adjusted. Converted output signals are directed to the Macintosh where maximum and minimum load values and the number of cycles are written to disc if the extreme values differ from previous values by more than a set amount (say, 1% full scale).

Tension and compression strength tests were carried out in the same servo-hydraulic testing machines as were used for the fatigue experiments at strain rates of approximately $1.5\times10^{-4}\ \mathrm{s}^{-1}$ in tension and $1\times10^{-4}\ \mathrm{s}^{-1}$ in compression. For compression tests anti-buckling guides of the kind described by Curtis (1988) were used. Elastic modulus values were obtained by means of clip-on strain gauges.

3. Experimental results

(*a*) *Mechanical properties*

The mechanical strength and stiffness characteristics of the four laminates are summarized in table 3. It can be seen that the tensile moduli of the T800/5245, T800/924 and IM7/977 materials are all alike, the average value being about 92 GPa, whereas that of the HTA/913 composite is only about 75% of this level. The IM7/977 composite, with a reinforcing fibre not unlike the T800, is closer to T800/924 in mechanical behaviour than to T800/5245.

By contrast, the compression strengths of the two T800 composites and the IM7 material show much more homogeneous behaviour. The HTA/913 material, surpris-

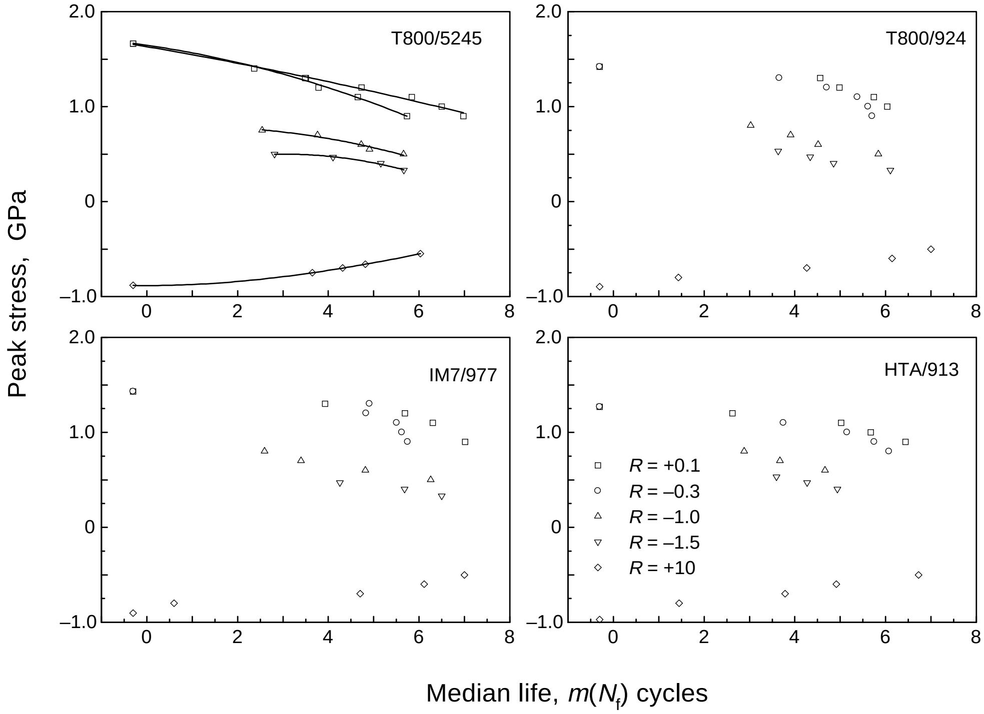

Figure 2. Stress–median-life data for four CFRP laminates of $[(\pm 45, 0_2)_2]_S$ construction at five R ratios. The data for T800/5245 are taken from Gathercole *et al.* (1994).

ingly, possesses a slightly higher compression strength than the other materials. Perhaps the most significant aspect of the results in table 3 is the fact that major differences in fibre tensile properties and matrix characteristics exert almost no influence on the compression resistance of these laminates.

(*b*) *Constant-stress fatigue response*

Replicate stress–life data for the three new laminates were obtained at the five R ratios +0.1, −0.3, −1.0, −1.5 and +10. In view of the consensus of opinion relating to the representation of data from stress–life tests in terms of the median fatigue lives, $m(N_f)$, rather than mean values (Johnson 1964; Little & Jebe 1975; Spindel & Haibach 1981; Young & Eckvall 1981), we have previously used the same approach and we continue to use it for the purpose of this analysis. Later in this paper, however, we give further thought to the significance of the median and other life parameters. The stress–median-life curves derived from the raw experimental data, including, for completeness, the results for T800/5245 from the previous publication, are plotted in figure 2. It is impracticable in a paper covering such a wide range of materials, R ratios and stress levels to include all test data, but some reference to scatter, etc., will be made later. The curves fitted through the data for the T800/5245 laminate are best-fit third-order polynomial curves and are shown as an indication of how data analysis may be carried out. The fitting of curves in this way does not indicate insight into the mechanisms of fatigue failure.

In our previous work with the $[(\pm 45, 0_2)_2]_S$ T800/5245 composite, we observed that the $\sigma/\log N_f$ data points for $R = +0.1$ and -0.3 appeared to fall on a curve that extrapolated smoothly back to the monotonic tensile strength of the laminate,

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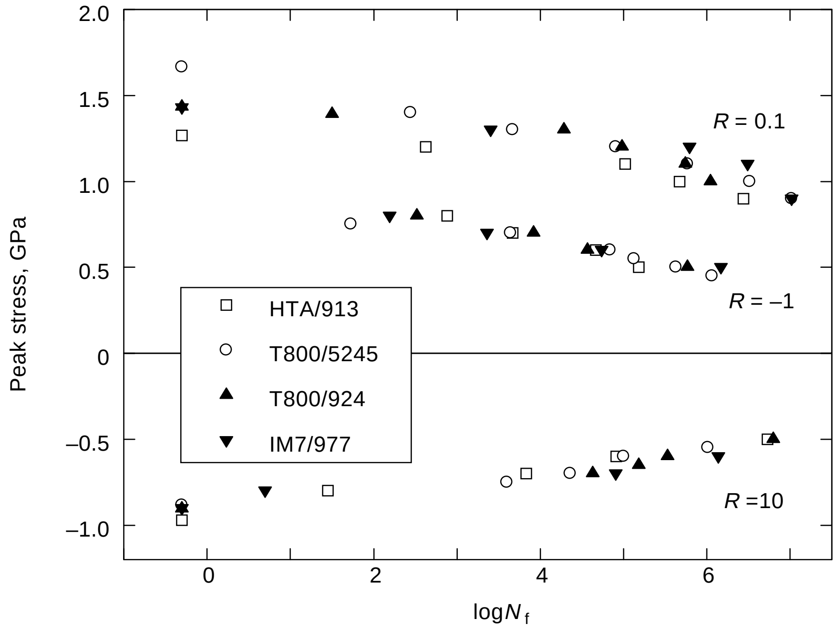

Figure 3. Comparison of fatigue results for the four CFRP laminates at three R ratios.

while those for $R = +10$ could be extrapolated smoothly back to the monotonic compression strength (measured in the same compression-test jig). It appeared, however, that the data points for R ratios of -1.0 and -1.5 were not associated in this way with the tensile strength, as shown by the curves drawn through those points for the T800/5245 laminate in figure 2. We concluded that the dominant mode of failure had effectively changed from being predominantly tensile as a result of the substantial compression stress component involved in cycling at $R = -1.0$ and -1.5, although it was not possible to find supporting microstructural evidence of the damage that could have been expected to accompany such a marked change. It appears from an examination of figure 2, however, that our interpretation of the T800/5245 results in isolation was hasty since the data points for all three of the newly investigated laminates at $R = -1.0$ and -1.5 may be extrapolated back to the tensile strengths of the laminate in question.

More direct comparisons of the four laminates may be made by replotting selected data. In figure 3, for example, are superposed the data for the four experimental laminates at R ratios $+0.1$, -1.0 and $+10$. It can be seen that the data sets for $R = +10$ and -1 for all four materials are largely overlapping and, although not shown in figure 3, the same is true for $R = -1.5$. Thus, it is only their behaviour under repeated tension cycling that distinguishes the fatigue response of these four materials, despite their obvious material differences.

4. Discussion

(*a*) *Statistical aspects of strength*

Weibull analysis reveals some further distinctions between the tensile strengths of the four laminates, as is discernible from the two-parameter Weibull plots shown in figure 4. The cumulative distribution function, F_W, represented in this figure is, of

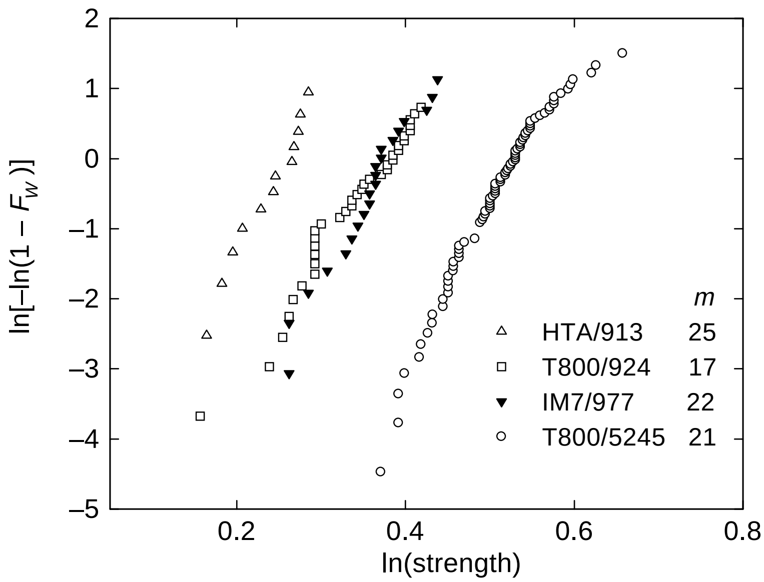

Figure 4. Two-parameter Weibull plots of tensile strength data for four CFRP laminates.

course, the familiar form of the type III model

$$F_{\mathrm{W}}(x; b, m) = 1 - \exp\left[-\left(\frac{x}{b}\right)^m\right], \tag{4.1}$$

where m is the shape parameter (or Weibull modulus) and b is the scale parameter. The shape parameters for the four data sets are all of the order of 20 and, although they show differences which suggest, for example, a higher level of variability in the T800/924 laminate than in the other three, inspection of the plots in figure 4 indicates that the differences may not be real. The data sets for T800/924 and IM7/977 are largely coincident and it can be seen that that the strengths of the T800/5245 and the HTA/913 are, respectively, significantly greater than and lower than those of the IM7/977 and the T800/924 composites. The rather surprising difference between the strengths of the two T800 materials must be due, it is supposed, to different levels of fibre/resin adhesion. The IM7/977 composite, with a reinforcing fibre not unlike the T800, is also apparently closer to T800/924 in mechanical behaviour than to the T800/5245.

It is frequently argued from a mechanistic point of view that the use of the two-parameter Weibull model (equation (4.1)) to represent data for mechanical strength must be invalid, since it makes the assumption that there is no finite stress level at which there is zero probability of failure, and that the only valid form of the statistical model must be the three-parameter model

$$F_{\mathrm{W}}(x; a, b, m) = 1 - \exp\left[-\left(\frac{x-a}{b}\right)^m\right], \tag{4.2}$$

where a is a location parameter. One can see the logic of the mechanistic argument, especially for a material such as one of these CFRP laminates which must, surely, exhibit a real zero-failure-probability stress (i.e. a finite value of the location parameter, a). The usual procedure in such cases is to make guesses at the value of a (perhaps starting at the minimum experimental value (see, for example, Chatfield

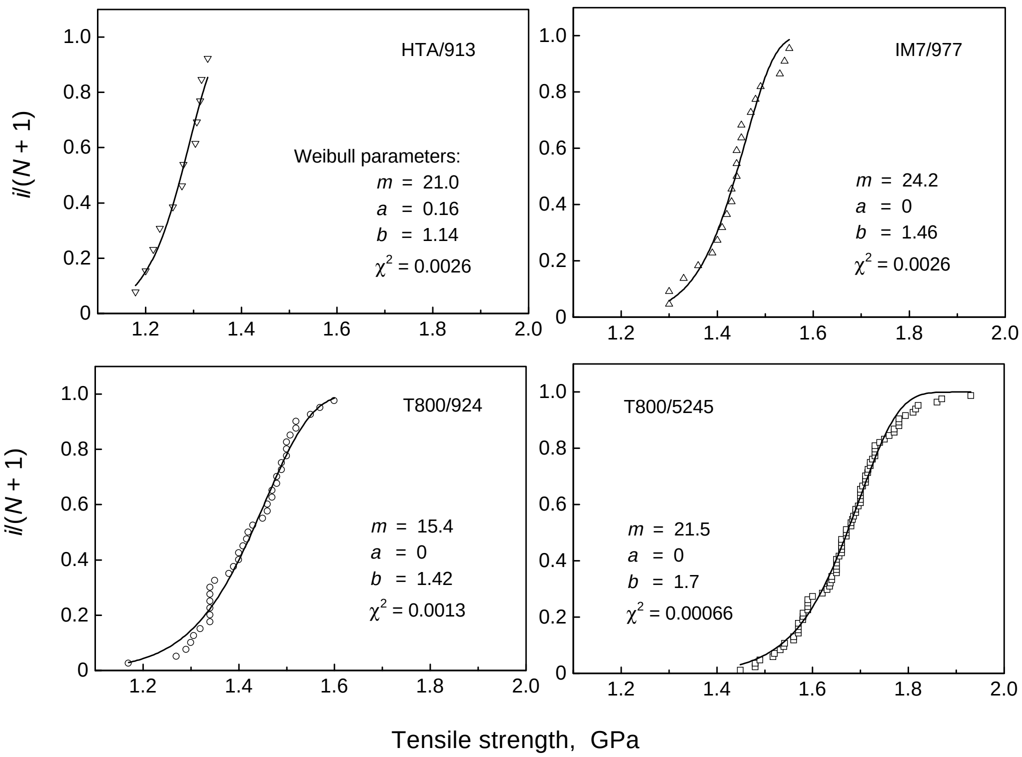

Figure 5. Three-parameter Weibull plots of the tensile strength data represented in figure 4.

1983)), taking $(x - a)$ as the transformed plotting variable in plots of the type in figure 4 until the best linear fit is obtained. An interesting alternative, however, is to use a nonlinear fitting procedure to obtain directly a best-fit curve for the relationship between the cumulative distribution function (CDF), F_W, and the measured parameter, in this case the tensile strength, together with appropriate values of all three distribution parameters, a, b and m. The results of figure 4 have been re-analysed in this way by means of the software package Origin, by Microcal. The straightforward mean-rank probability method† is used to derive the order statistics of the CDF, i.e.

$$F_W = i/(N + 1), \tag{4.3}$$

where i is the rank number and N is the number of data values. The nonlinear curve-fitting process uses the Levenberg–Marquardt algorithm (Press *et al.* 1988) after an initial application of a Simplex minimization (based on the method of Nelder & Mead (1965)) for parameter initialization. The fitting results are shown in figure 5, in which the fitting parameters m, a and b are given on the diagram, together with the value of χ^2 returned by the programme as the indicator of goodness-of-fit. It is somewhat surprising to note that in three out of the four cases, the value of the location parameter, a, turns out to be zero and for the fourth case nearly zero, despite the unlikelihood that physical failure of any of these laminates could ever occur at stresses much below the minimum test strengths shown in the diagrams. Despite the physical argument, therefore, it does not seem so unreasonable to use

† Brief consideration of the choice of ranking method is given in the appendix.

Table 4. *Weibull parameters for the strengths of the experimental CFRP laminates*

	Weibull model and parameter (para.)					
material	2-para. m	3-para. m	2-para. b	3-para. b	2-para. a	3-para. a
tension						
T800/5254	21.4	21.5	1.71	1.70	0	0
T800/924	17.0	15.4	1.46	1.42	0	0
IM7/977	21.9	21.0	1.46	1.46	0	0
HTA/913	25.1	24.2	1.29	1.14	0	0.16
compression						
T800/5254	9.9	9.1	0.93	0.92	0	0
T800/924	10.1	15.9	0.94	0.87	0	0
IM7/977	12.3	11.6	0.94	0.94	0	0
HTA/913	12.4	14.5	1.01	1.00	0	0

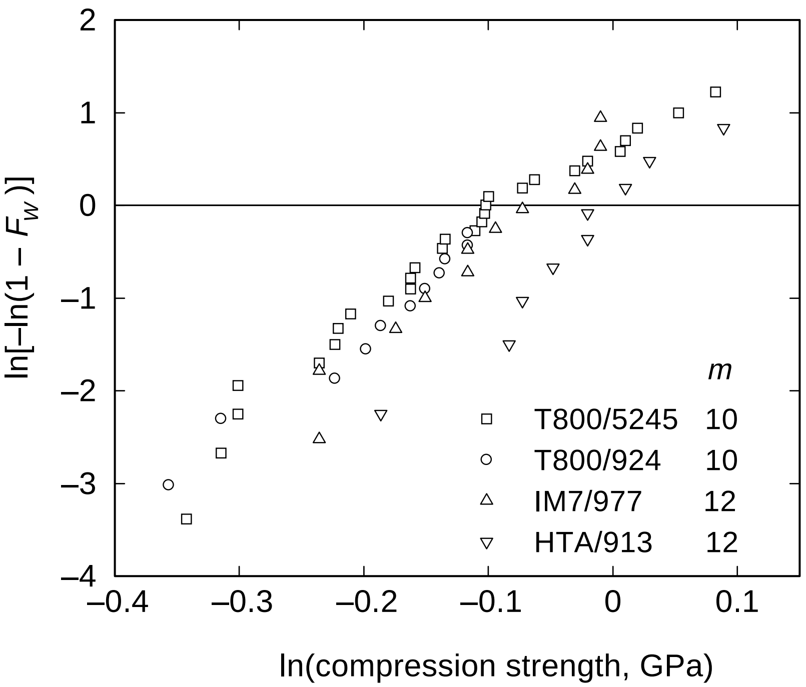

Figure 6. Two-parameter Weibull plots of compression strength data for four CFRP laminates.

the two-parameter Weibull model, even for materials such as these laminates. The values of the Weibull parameters obtained from the two methods may be compared in table 4 and it can be seen that the level of agreement is extremely high.

By contrast with the tensile strengths, the compression strengths of the two T800 composites and the IM7 material show much more homogeneous behaviour, as can be seen from the two-parameter Weibull plots in figure 6, while the HTA/913 material, surprisingly, shows a significantly higher strength than the other materials. The level of variability is similar in all four materials, since the Weibull shape parameters are all in the range 10–12. The plots are all reasonably linear, despite the much smaller numbers of test results. It is interesting to note that nonlinear curve fitting to the three-parameter Weibull model produces a result similar to that discussed above for the tensile strengths, namely that the fitted location parameter, a, is zero for all four laminates and the predicted values of m and b are again similar to those obtained from the linear curve fit. The comparison is again made in table 4 and it can be seen that, in this case, the smaller numbers of test data lead to slightly poorer agreement in the case of two of the composites.

(*b*) *Statistical analysis of fatigue data*

It is appropriate at this point to give some thought to the statistical analysis of fatigue data with specific reference to two separate issues: (i) the selection of an appropriate design-life parameter (as touched on earlier); and (ii) the problem of obtaining safe design criteria for applications involving composite materials from relatively small experimental data sets. Since the acquisition of fatigue data for any new material is costly and time-consuming, and since new composite materials are constantly being introduced, it is of advantage to the designer to have available some conservative means of estimating the potential usefulness of such a new material as soon as possible. Thus, although it is a *sine qua non* that before any new material is put into real service its fatigue properties must be exhaustively investigated, there is considerable value in having available some means of judging the likely fatigue performance from a relatively small data bank. This is not as hazardous as it sounds, despite what we know of statistical methods of analysis, because of the regular patterns of behaviour that are appearing from current research.

(i) *Statistical concepts and extreme-value theory: the Weibull model*

Extreme-value models are appropriate models for describing many engineering phenomena. Typical examples are given by Bury (1975) for systems where the relevant parameters are the characteristic largest and characteristic smallest values of a distribution. For example, in a system of many parallel components, the lives of which are variable, it is the life of the longest-lasting component which determines the life of the system as a whole. In such a case, it is the type I model for maxima which is relevant, whereas, by contrast, in the case of the life of a gas-turbine disc fitted with a large number of blades, the turbine life is determined by the life of the shortest-lived blade—a 'weakest-link' model. Such a model, referred to as a type III asymptote model, is also applicable to the fatigue behaviour of engineering materials. It is useful at this stage to review the nature of the Weibull model and to indicate the use that has been made of it by other researchers.

Let x_1 be the minimum extreme value of n measurements of X in a model identified by the formula $f(x;\theta)$, which is known to be bounded at its lower limit by some threshold value a (the parameter a may often be zero in engineering problems). What has come to be known as the Weibull model may be stated in terms of f_{W}, the probability distribution function (PDF), in the form given by Freudenthal &

Gumbel (1954)

$$f_{\mathrm{W}}(x;a,b,m)=\frac{m}{b'-a}\left(\frac{x-a}{b'-a}\right)^{m-1}\exp\left[-\left(\frac{x-a}{b'-a}\right)^{m}\right],\quad 0<(x-a),(b'-a),m. \tag{4.4}$$

In equation (4.1), the PDF is presented in terms of the three-parameter Weibull model for minima, where the symbols m, a and b were defined in § 4 a. In this form, the scale parameter, b, is given explicitly as $(b'-a)$ and the form of equation (4.4) is thus somewhat more logical than the normal form of the Weibull model, in which b appears to vary when the location parameter is changed, whereas b' remains invariant. Since a transformation of the form

$$\frac{\alpha}{\beta}=\frac{x-a}{b'-a}$$

converts the three-parameter model to a two-parameter model (i.e. the situation where a is known (see Castillo 1988, p. 199)), it is convenient to begin by considering the simpler form

$$f_{\mathrm{W}}(x;b,m)=\frac{m}{b}\left(\frac{x}{b}\right)^{m-1}\exp\left[-\left(\frac{x}{b}\right)^{m}\right],\quad 0<x,b,m. \tag{4.5}$$

This is the type III asymptote of the minimum extreme value among measurements modelled by an initial distribution which is bounded below (i.e. $a\geqslant 0$). The cumulative distribution function (CDF) corresponding to f_{W}, represented by F_{W}, is given by

$$F_{\mathrm{W}}(x;b,m)=\int_{x=0}^{x}f_{\mathrm{W}}(x;b,m)\,\mathrm{d}x \tag{4.6}$$

or

$$F_{\mathrm{W}}(x;b,m)=1-\exp\left[-\left(\frac{x}{b}\right)^{m}\right], \tag{4.7}$$

which is the form introduced earlier as equation (4.1).

The characteristic value, $v_{\min}$, of an initial sample, X, is the value x for which there is, on average, only one smaller observation among n (following Bury (q.v.)). Thus,

$$nF(v_{\min};\theta)=1,\quad F(v_{\min};\theta)=1/n,\quad \text{i.e. } v_{\min}=\text{quantile } q=(1/n).$$

Thus, $v_{\min}$ is the quantile of order $1/n$ and it decreases as n increases. Since $F(x;\theta)$ is bounded at $a\geqslant 0$, $v_{\min}$ approaches a as the number of samples, n, increases.

The exact CDF of the smallest extreme value, x_1, of an initial sample X is given by

$$\varphi_1(x)=1-[1-F(x;\theta)]^{n}. \tag{4.8}$$

When $x=v_{\min}$, $F(v_{\min};\theta)=1/n$ and

$$\varphi_1(v_{\min})=1-[1-(1/n)]^{n}. \tag{4.9}$$

As the initial sample size, n, increases, φ_1 approaches $[1-\exp(-1)]$, which is thus the asymptotic probability of observing a minimum $v_{\min}$. But, if $F(x;\theta)$ is bounded below, this probability is

$$F_{\mathrm{W}}(v_{\min};b,m)=1-\exp(-1),$$

and this represents the Weibull CDF when $b = v_{\min}$. Thus, the characteristic value of the type III asymptote is equal to the Weibull scale parameter, b. Since $v_{\min}$ is a function of the initial sample size (equation (4.9)), the scale parameter, b, will change with n when F_{W} is applied as an extreme-value model.

Some of the familiar parameters of a Weibull PDF are

(i) the expected value of x (first moment of the distribution, or arithmetic mean)

$$b\Gamma\left(1+\frac{1}{m}\right),$$

(ii) the mode (most probable value)

$$b\left[\frac{m-1}{m}\right]^{1/m},$$

(iii) the median

$$b(\ln 2)^{1/m},$$

(iv) the variance (second moment of the distribution)

$$b^2\left[\Gamma\left(1+\frac{2}{m}\right)-\Gamma^2\left(1+\frac{1}{m}\right)\right],$$

(v) the general quantile of order q

$$b\left[\ln\left(\frac{1}{1-q}\right)\right]^{1/m},$$

(vi) the characteristic value of x (i.e. the 0.63 quantile)

$$b$$

and Γ is the gamma function.

An important aspect of the Weibull distribution is its reproductive property with respect to its own minimum value. The exact distribution of the smallest observation in a Weibull distribution is again a Weibull model. Thus, as demonstrated by Epstein (1948) and by Bury (q.v.), the function of equation (4.8)

$$\varphi_1(x;b',m') = 1-[1-F(x;b,m)]^n = 1-\left\{\exp\left[-\left(\frac{x}{b}\right)^m\right]\right\}^n$$
$$= 1-\exp\left[-\left(\frac{x}{b/n^{1/m}}\right)^m\right], \quad \text{i.e. } \varphi_1(x;b',m') = F_{\mathrm{W}}\left(x;\frac{b}{n^{1/m}},m\right). \quad (4.10)$$

Hence, if a sample of size n is taken from a box modelled by a Weibull function, the smallest observation, x_1, exhibits a Weibull distribution rescaled by $n^{1/m}$. The characteristic minimum value for a test sample of n tests will therefore be $(b/n^{1/m})$ and the modal (most probable) value will be given by

$$\frac{b}{n^{1/m}}\left(1-\frac{1}{m}\right)^{1/m}.$$

The shape parameter of the new distribution, which we might designate m^*, remains the same ($m^* = m$).

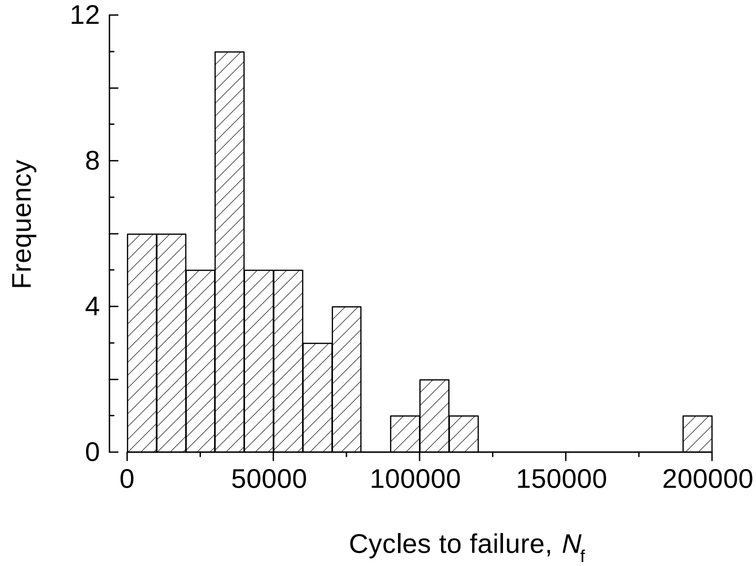

Figure 7. Histogram of fatigue lives for ud HTA/982 laminate at a peak stress of 1.6 GPa ($R = 0.1$).

(ii) *Application to results of fatigue tests on a unidirectional HTA/982 CFRP laminate*

One of the problems in designing with composites for a fatigue environment is that of knowing how to cope with the variability of fatigue test data, which can be large in fibre composite materials. As an example, we consider the histogram of figure 7, which is for a replicate set of 50 fatigue tests on a unidirectional CFRP laminate of the system HTA/982. The tensile strength of this material is 2.1 GPa and the fatigue tests were carried out in repeated tension ($R = +0.1$) at a peak stress of 1.6 GPa (i.e. 76% of the tensile strength).

The difficulty is that the experimental data cover a great range of sample lives. The greatest number of failures occur in the band 30 000–40 000 cycles, but the shortest observed life was 1392 cycles and there is a single outlier at nearly 200 000 cycles. Conventional design procedure would be to take the median value or some other required failure probability and use a substantial safety factor. Choice of the smallest observed value of life would clearly be equivalent to the use of an impossibly large safety factor, and yet the CDF of the data set represented in figure 7 suggests that there is no minimum life for this material at this stress level—a possibility that would certainly disturb users of fibre composites.

In carrying out fatigue tests on composite materials, we usually try to obtain safe stress–life curves with as few samples as possible because of the cost of extensive testing programmes. In a replicate series of tests at several stress levels, the variances of the lives at each separate stress level will usually be of the same order of magnitude and this often gives us sufficient confidence to reduce the number of individual replicates at each stress level. One of the problems is to know how many replicate tests should be done at each stress level since, from an economic point of view, the smaller the number of tests that can be used to establish a 'safe' $\sigma/\log N_f$ curve, the better. It is commonly accepted that at least 20 individual tests may be necessary before the user can have any confidence in a statistical analysis of results and King (1989) suggests that the minimum number may be even higher (30–45, depending on the type of composite). Yet when stress–life curves are required at, say, five different R ratios, even five tests at each stress level may be all that can be provided in a reasonable amount of time, especially at long lives. It has been demonstrated (see, for example, Whitney 1981; Gathercole *et al.* 1984) that pooling of data from many stress levels and R ratios is permissible and that the pooled data may be used to obtain a Weibull shape parameter which may then be used with confidence to determine appropriate design allowables. It is interesting to speculate, however, on the likelihood of obtaining, for a material such as that represented in figure 7, test lives as low as 1392 cycles in a test series where n is much smaller than 50.

This may be explored with the aid of the model described earlier. If we assume that the 50-sample data set reasonably represents the exact distribution of the fatigue lives at the chosen stress level, $F_{\mathrm{W}}(x; b, m)$, we may consider a small number, n, taken at random from the full data set, to represent a test set of the more limited kind from which we would usually construct our stress–life design curve. The theory presented above suggests that the distribution of the minima of several such samples should also be Weibull, with a scale parameter of $(b/n^{1/m})$ and a shape parameter m. In order to test this, a random-number program was used to select 20 samples of seven test results each, from the full set, and the minima of these sub-sets were fitted to a Weibull model.

There is disagreement about which of the several possible methods available for determining the parameters of a Weibull distribution is the most appropriate. Freudenthal & Gumbel (1953, 1954) demonstrated the use of the method of moments proposed by Weibull (1949), but King (1989) observes that it is not a very efficient method and is not used for composite materials. Whitney (1981) used the maximum likelihood estimate (MLE), considered to be superior to estimates derived by means of linear regression methods (Shooman 1968), but Castillo (1988) suggests that because of the peculiar behaviour of the Weibull distribution for $1 < m < 2$, the MLE is not appropriate for Weibull distributions with $m < 2$, although it is highly recommended elsewhere because it gives the lowest variability. It is interesting to note that the m values reported by Whitney and by Gathercole *et al.*, for very different materials, were both close to unity, whereas for the fatigue lives of metallic materials, $2 < m < 6$, as pointed out by Freudenthal & Gumbel (1953). Talreja (1981) also rejected the MLE approach in favour of a standardized variable estimation method originally proposed by Lieblein (1955), suggesting a lower limiting value of m of about 10 for valid use of the maximum likelihood method.

The choice of one method of estimating the distribution parameters over any other may be made on the basis of the relative variances of the predicted values or (less frequently now in view of the ready availability of high-speed computers) on the basis of the labour involved in calculations. Although statisticians agree that the simplest method of estimating the distribution parameters, namely the linear least-squares technique, is unacceptable because, as Bury (q.v.) observes, order statistics are dependent, it is curious to the non-specialist that fitting a straight line by eye to a plot of $\ln(-\ln(1-F_{\mathrm{W}}))$ versus $\ln x$ to obtain estimates of m and b has authoritative support (see, for example, Weibull 1961; Gumbel 1958; Chatfield 1983). For the purposes of this initial analysis, we shall use the linear regression method.

Figure 8 shows conventional two-parameter Weibull plots of the full data set represented in figure 7 and the minimum-life data set obtained by random selection.

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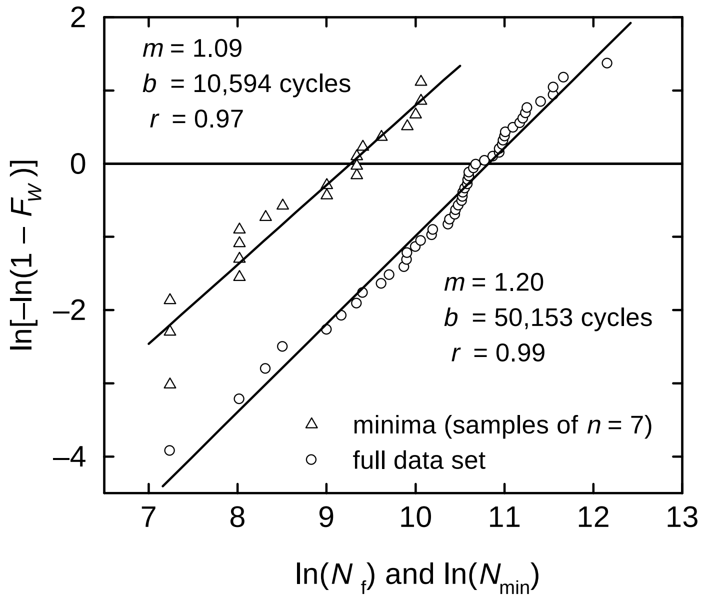

Figure 8. Two-parameter Weibull plots for a unidirectional HTA/982 CFRP laminate. The lower data set is a complete set of 50 test results and the upper set represents the minima of 20 groups of seven results selected at random from the full data set ($R = +0.1$).

The mean-rank probability method (equation (4.3)) was again used to derive the order statistics of the CDF. The degree of linearity in both cases is high and it can be seen that the shape parameters are both close to unity, the regression lines being almost parallel. The fitting algorithm in the Origin software gives values for the two slopes, m_1 and m_2, of 1.202 (0.025) and 1.086 (0.068), the figures in brackets being the standard deviations of the slopes. The t statistic for the difference is 2.339 for 70 degrees of freedom and, since the tabulated value of t for 70 degrees of freedom is 2.381 at the 0.02 level, even this small difference in slopes is therefore just about significant, statistically speaking, although for practical purposes it is not important. The scale parameter for the 'complete' population is 50,153 cycles, while that for the distribution of minima is 10,594 cycles.

The distribution parameters indicated on figure 8 may now be used to plot the normalized PDFs for the two data sets, as shown in figure 9. Also plotted in figure 9 is the curve defined by equation (4.10) with the same value of m as the full data set and the modified scale parameter, $b_{\text{sample}} = b/m^{1/n} = 50\,153/(1.2)^{1/7}$, or 48 864 cycles. It can be seen that on this linear plot, the two curves defining the PDFs for minima are not very different in the neighbourhood of the peak values. The modal values for the minima are 1065 and 1925 cycles, respectively, for the 'experimental' and 'theoretical' curves, a difference of no importance given the nature of the distribution shown in figure 7. The modal value of life for the full data set is 11 268 cycles.

The parameters of the distribution of fatigue lives for the laminate are: (i) expected value of x (arithmetic mean, $\sum x/n$), 44 682 cycles; (ii) median, 38 000 cycles; (iii) characteristic value (b), 50 153 cycles; (iv) shape parameter, m (by linear regression of y on x), 1.202.

Expressions (i) and (iv) in § 4 b (i) for the first two moments of the distribution (mean and variance) may also be solved as a pair of simultaneous equations to find m and b (method of moments) from the experimental distribution parameters; this

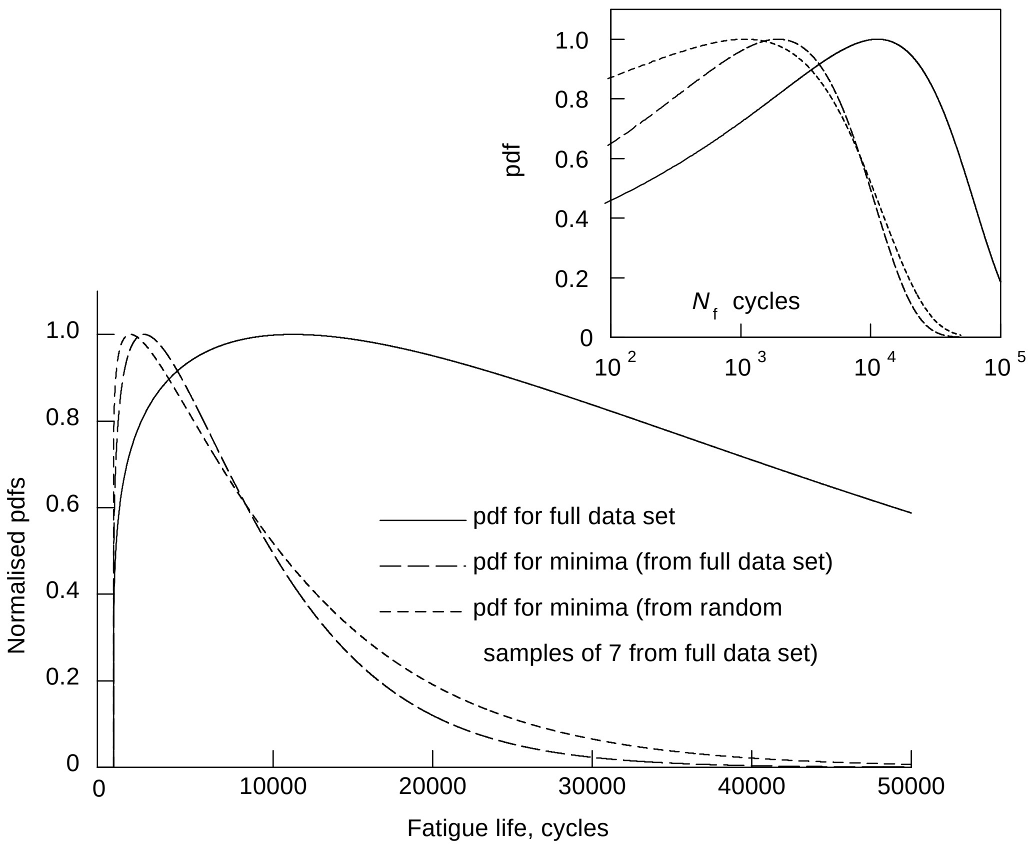

Figure 9. Probability density functions for a unidirectional HTA/982 CFRP laminate. The full curve is for a complete set of 50 test results. The broken curves represent distributions of minima, one being predicted from the Weibull parameters of the full data set and the other an 'experimental' PDF for the minima of 20 groups of seven results selected at random from the full data set ($R = +0.1$).

yields values of $m = 1.247$ and $b = 46\,100$ cycles. Thus, the value of m predicted by this method is to all intents and purposes the same as that obtained by linear regression and the value of b is sufficiently close to the linear regression value of 50 153 for design purposes. The median life predicted from these distribution parameters, $b(\ln 2)^{1/m}$, is 36 593 cycles.

With reference to the comment made earlier about the validity of using the simple linear regression method of estimating m, it seems reasonable to suppose that since fitting a straight line by eye intuitively involves minimizing errors in both x and y directions rather than in the y direction alone as in the normal least-squares procedure, a more appropriate way of estimating m from the straight lines in figure 8 would be to take the geometric mean, $\sqrt{(m'm'')}$, of the slopes of the two regression lines which describe the linear functions $\hat{y} = m'x + c_1$ and $y = m'' + c_2$, which are obtained by minimizing the sums of the squares first in the y direction and then in the x direction. When the distribution in question is well represented by a straight line (correlation coefficient $\rightarrow 1$), the two slopes will coincide, but not otherwise. For the laminate data in figure 8, the regression line for x on y has the slope 0.815, of which the reciprocal is 1.227, and the geometric mean value of the two slopes is therefore 1.215. In this case, then, the difference is insignificant.

(iii) *Application to constant-life fatigue data for* $[(\pm 45, 0_2)_2]_S$ *laminates*

The analysis described in the previous section dealt with the special case of a single data set representing the distribution of fatigue lives at a one stress level. The assumption is that a set of 50 test results is sufficiently large to be considered to represent the complete population. This is likely to be an invalid assumption from a statistical point of view, but it is clearly more valid to make predictions from 50 results than from, say, 10, when dealing with a property as variable as the fatigue response of a composite. Even so, it is scarcely a practical or economic matter for a designer to demand a family of fatigue $\sigma/\log N_f$ curves based on 50 or so replicate tests at each stress level. Thus, the data pooling method used by Whitney (1981) has considerable advantages. Whitney suggested that where small numbers of stress–life values are available at a number of different stress levels, the data may be pooled to give an overall value of the Weibull shape parameter, m, this value then being used to obtain working stress–loglife curves for any given failure probability. This is done by normalizing each group of data with respect to the characteristic life (the Weibull scale parameter), pooling the data and then re-ranking them in order to allot a new failure probability function to each point. In our earlier paper (Gathercole *et al.* 1984), we applied the method to our results for the T800/5245 laminate and found, like Whitney, that the single pooled data set showed a very high degree of uniformity, with a shape parameter, m, of 1.1, almost identical with that obtained by Whitney, quoting data by Ryder *et al.* (1977) for a similar type of carbon-fibre composite. This analysis of the stress–life data was confirmed by Lamela-Rey (1994) in a more rigorous analysis based on extreme-value theory by Castillo *et al.* (1993).

An interesting question, in relation to the pooling method, is whether the use of the scale parameter for normalization is a valid method. Where only a few data points per stress level are available, the application of the least-squares method to analyse a two-parameter Weibull plot can produce highly variable values of m (the slope of the curve), as shown in our earlier paper, although the b values appear to be more consistent because they are indicators of central tendency, whereas m values are more affected by extreme values. For this reason, we have modified our approach for the present analysis by normalizing the data sets for each individual stress level with respect to the actual median, $m(N_f)$, of the group. This also permits the use of data sets consisting of only two results, although the majority of data sets consist of many more than this. The pooled, normalized life data obtained from the raw test results from all stress levels and R ratios for all four CFRP laminates are shown on two-parameter Weibull plots in figure 10. Although the correlation coefficients for the fits are all high, we note that this method of normalization produces pooled data sets which fall less satisfactorily on straight lines than was the case; for example, for the same T800/5245 data plotted in the paper by Gathercole *et al.* (1984) where the normalization was with respect to the characteristic value, as discussed above. This is presumably because the latter method produces an element of smoothing, whereas when the median is used for the normalization, a high proportion of the normalized data points will have a value of unity, hence the step which appears at an abscissa value of about zero in all four graphs. In each graph, there is also a small number of abnormally long lives which result in a slight flattening of the data distribution. These points do not fit well on the regression lines, but, as pointed out by Castillo (q.v. p. 203), the regression line must in any case be fitted to data in the tail of interest because the Weibull distribution is an asymptotic distribution and generally gives an approximation for that tail only; in this instance the lower end

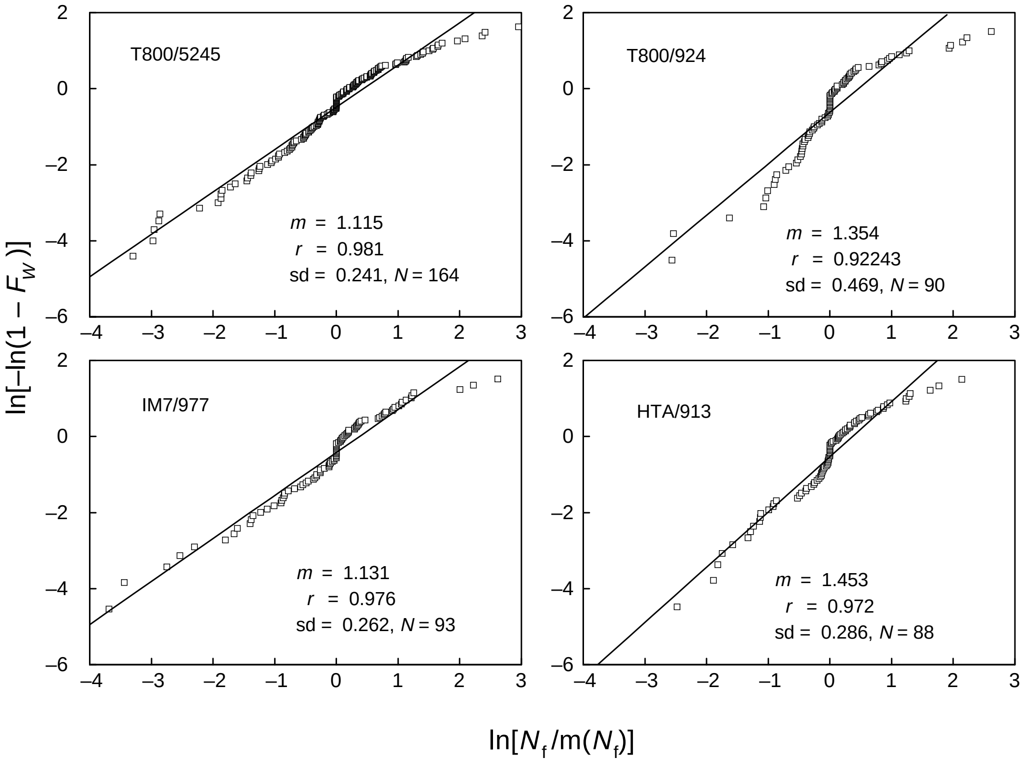

Figure 10. Two-parameter Weibull plots of pooled normalized fatigue lives for all stress levels and R ratios. The lives are normalized with respect to the median life, $m(N_{\mathrm{f}})$, in each individual data set.

of the distribution. This is also an argument for the safe ignoring of run-out data. We note that, of the four materials, only the data for the T800/924 composite give a line which is less than convincing and the m value obtained for the T800/5245 is substantially the same as that reported in our earlier paper, namely 1.1.

A small number of points in each of the pooled data sets appeared to represent failures at abnormally low lives and, because of the logarithmic scale, these distorted the linearity of the plots considerably. Inspection of the data sets revealed in each case that these points were for lives in repeated tension or tension–compression that were less than 100 cycles and almost always two orders of magnitude or more than other lives in the set. Since the servo-hydraulic machines rarely settle to the preset cycling pattern in less than 100 cycles (less than a half minute), it seems reasonable to ignore these points.

For the sake of completeness, values of the Weibull parameters obtained by a number of different procedures are compared in table 5. In each group of the table, the first two estimates are the linear (two-parameter Weibull model) and nonlinear (three-parameter Weibull model) estimates for the pooled data normalized with respect to the experimental median life and the third is a nonlinear estimate based on data normalized, as in the Whitney model, with respect to the characteristic life, b, for the individual data set, regardless of how small the data set is. The fourth estimate in each group was obtained by the method of moments, as described in § 4 *b* (ii), by solving the two simultaneous equations defining the arithmetic mean and variance in terms of the Weibull parameters. The differences between values of

Table 5. *Comparison of Weibull parameters for distributions of fatigue lives, obtained by different methods, for four CFRP laminates*

model	normalization method	global para.	material T800/5254	T800/924	IM7/977	HTA/913
linear regression	median, $m(N_f)$	m	1.12	1.35	1.13	1.45
nonlinear	median, $m(N_f)$	m	1.25	1.49	1.41	1.71
nonlinear	b, for each data set	m	1.13	1.34	1.10	1.53
moments (linear)	median, $m(N_f)$	m	0.74	0.78	0.78	1.07
linear regression	median, $m(N_f)$	b	1.55	1.58	1.45	1.44
nonlinear	median, $m(N_f)$	b	1.24	0.94	1.23	1.17
nonlinear	b, for each data set	b	0.93	0.85	0.92	0.93
moments (linear)	median, $m(N_f)$	b	1.32	1.34	1.24	1.37
linear regression	median, $m(N_f)$	a				
nonlinear	median, $m(N_f)$	a	0.09	0.03	0.05	0.1
nonlinear	b, for each data set	a	0	0.02	0	0
moments (linear)	median, $m(N_f)$	a				

m obtained by the first three methods are very small and the choice of a method for the practical purpose of life prediction would represent no difficulty. The values of b given by the three methods are much more variable but, since pooling is carried out only to provide a reliable value of the shape parameter, this variability is immaterial. The formal statistical method of moments gives values of m which differ significantly from those given by the other methods, three out of the four being less than unity which precludes the use of any further analysis that involves the calculation of a mode. Even the value for T800/5245, which has the largest data set and shows the best straight-line fit to the pooled data, is very much lower than the values given by the other procedures. This standard method appears not to be satisfactory, therefore, by contrast with the findings of Freudenthal & Gumbel (1953, 1954) for metallic materials.

The pooling concept makes the assumption that the shape parameter for fatigue life distributions is not a function of the cyclic stress. There is *a priori* no justification for making this assumption and indeed it has been suggested by Freudenthal & Gumbel (1956) that, on the contrary, m will be a function of stress. In our previous paper, however, we showed that, for the T800/5245 laminate, the values of m determined for small samples at all R ratios varied widely between about 0.3 and 3.0 but showed no evidence of any consistent variation with stress. Nakayasu (1987) has shown that the Weibull shape parameters for a number of related fatigue-life data sets may themselves be modelled by a Weibull distribution. For a large number of data sets for a medium-carbon steel, for example, he obtained a good fit to a three-parameter Weibull model with the median value of the shape parameter being 1.69. If all of the individual data sets for all four laminates under consideration in this paper are treated in the same fashion, we obtain similar results, with the shape parameters for the high-performance fibre laminates all being close to unity, while that for the HTA/913 material is slightly higher, at 1.37. The location parameters for the materials correspond roughly to the lowest estimated m value in each case, between 0.2 and 0.3, while the scale parameters are between 0.7 and 1.1.

Having established a reasonable value of the shape parameter, m, from a pooled data set, the designer then has access to a range of design parameters beyond the usual mean or median through the relationships listed in § 4 *b* (i). It is interesting, as an example, to compare some of these parameters for a particular set of stress–life data, namely the $\sigma/\log N_{\mathrm{f}}$ curve for the T800/5245 laminate at $R = 0.1$, for which the numbers of replicate tests for the six available stress levels vary between 5 and 12. Figure 11 shows the relationships between some of these parameters. The curve to the extreme right is a $\sigma/\log N_{\mathrm{f}}$ curve based on arithmetic means. The two sets of triangles represent medians, the experimental medians being the upright triangles, while the inverted symbols are obtained from the distribution parameters as $b(\ln 2)^{1/m}$, m being obtained from the pooled data set. On the logarithmic plot, there is no significant difference between the two and the dashed curve is drawn through the averages of the pairs of values. The diamond-shaped points represent the modal values of the PDFs, $b(1-(1/m))^{1/m}$, and the open squares represent the modal values of the distributions of the minima, as discussed in § 4 *b* (i). This extreme left-hand curve is effectively a reasonable estimate of the most likely minimum lives at each of the stresses represented and is a possible alternative to the usual type of quantile or percentage probability failure curve. It is interesting to note that the scale parameters, b, for the individual replicate data sets, deduced with the aid of the predetermined m value, are indistinguishable from the arithmetic mean values in figure 11. The curves drawn through the several sets of points are third-order polynomials and it can be seen that they form a family in which the separation of the minimum and the mean is roughly a constant $1\frac{1}{2}$ decades over the stress–life window represented. It is clear that, provided there is an awareness of the relative positions of these families of curves, there is no particular justification in insisting, for example, on the use of the median as opposed to the mean in presenting fatigue data.

It is interesting that the minimum-life curve in figure 11 is smooth and closely in register with the experimental median-life curve, despite the fact that the numbers, n, of replicate test results at the different stress levels are small and vary considerably. The sensitivity of the minimum-life curve to the value of the Weibull shape parameter, m, and to the size of the replicate test sample may be demonstrated, as shown in figure 12, by plotting the scaling factor, $(n^{1/m})^{-1}$, derived from equation (4.10), as a function of n for selected m values. It can be seen that for a representative m value of 1.3, the scaling factor falls from about 0.25 to about 0.17 when the sample size is doubled from 5 to 10. Similarly, for a fixed sample size of 5, the factor falls from 0.33 to 0.23 when m falls from 1.5 to 1.1. Both of these reductions imply an improved level of confidence, since they involve a shift of the minimum-life curve to the left, but on a logarithmic life scale the shifts would not be particularly dramatic.

(*c*) *Constant life analysis and life prediction*

The median-life data of figure 2 and figure 3 are plotted in terms of peak stress as a function of life. It is apparent, however, that, as the compression component of cycling increases (R increasingly negative), the stress range, $2\sigma_{\mathrm{alt}}$, to which the sample is subjected initially, increases. If the data were plotted as stress range versus loglife, the data points for $R = -0.3$ would then lie above those for $R = +0.1$. This

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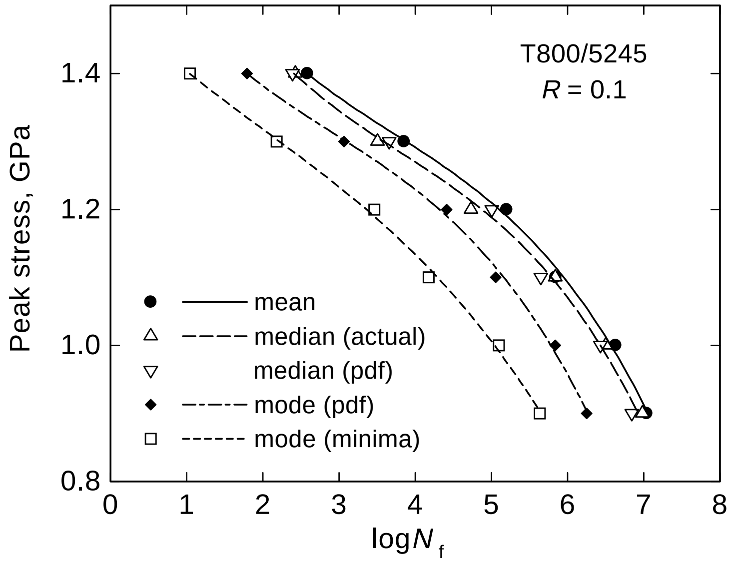

Figure 11. Stress–log-life curves for $[(\pm 45, 0_2)_2]_S$ T800/5245 laminate at $R = 0.1$. The fitted curves are third-order polynomials. The curve for the median data is plotted through the averages of the two points shown (actual and PDF values). The sample b values are indistinguishable from the mean lives on this logarithmic plot.

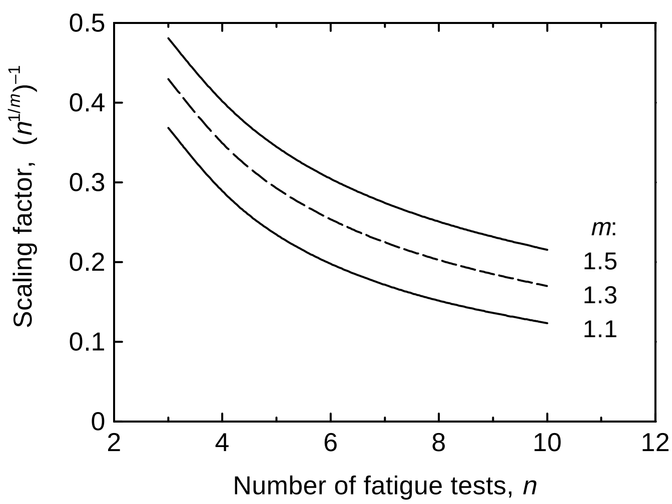

Figure 12. Sensitivity of extreme-value b parameter to the value of the Weibull modulus, m, and the number of test results, n. Data for T800/5245 $[(\pm 45, 0_2)_2]_S$ laminate at $R = +0.1$.

is a familiar feature of fatigue in fibre composites and it indicates that some element of compression load in the cycle can apparently improve the fatigue response. It also results in a well-known aspect of composites fatigue, namely that master diagrams of the constant life, or Goodman, variety are displaced from symmetry about the alternating-stress axis at $R = -1$ (Schütz & Gerharz 1977; Howe & Owen 1972; Kim 1988). In some of our earlier work (Adam *et al.* 1986, 1989; Fernando *et al.* al 1988), we showed that, for a family of hybrid carbon/Kevlar composites, the effects of R ratio could be illustrated by presenting the fatigue data as a normalized constant life diagram by means of the fatigue parameter, f

$$f = \frac{a}{(1-m)(c+m)}, \tag{4.11}$$

where $a = \sigma_{\text{alt}}/\sigma_{\text{t}}$, $m = \sigma_{\text{m}}/\sigma_{\text{t}}$ and $c = \sigma_{\text{c}}/\sigma_{\text{t}}$. σ_{alt} is the alternating component of stress, which is equal to $\frac{1}{2}(\sigma_{\max} - \sigma_{\min})$, and m is the mean stress, $\frac{1}{2}(\sigma_{\max} + \sigma_{\min})$. In these definitions, σ_{t}, σ_{c} are the monotonic tensile and compressive strengths, respectively. For the purposes of this parametric analysis, we keep the sign of σ_{c} positive, so that the parameter c is also positive. The stress function, f, depends on the test material. Since this is a parabolic function, the criterion is more akin to the old Gerber function (Gerber 1874) than the linear Goodman relationship (Goodman 1899).

Although these and other early results fitted the simple parabolic model for a particular life reasonably well, as more R values were investigated for more materials it became increasingly apparent that a more complex function was required. A bell-shaped curve seemed more representative and it could be seen that equation (4.11) was a special case of the more general function

$$a = f(1-m)^u(c+m)^v. \tag{4.12}$$

In the first instance, the development of this model was purely empirical; there was no *a priori* reason to suppose that the values of the parameters f, u and v had any special significance relative to the material structure/properties relationship or the local fatigue damage mechanisms contributing to failure.

(i) *Application of the model for life prediction*

In this part of our paper, we describe the stages in the analysis, illustrated with results for various materials.

Task 1: data acquisition. In addition to determining the tension and compression strengths of any new material, it is necessary to obtain stress–life data for a series of stress levels that cover to an adequate extent the whole working range from the monotonic strength level down to any notional 'endurance limit'. Initially, five or so replicate tests at each of four or five stress levels will define a reasonable stress–life curve for a particular R ratio and data for several R ratios from repeated tension to repeated compression will be required. Such small numbers of tests are not in themselves adequate for defining a true probability distribution function at any stress level, but the multiplicity of stress levels increases the confidence in fitting a curve, whether it be a polynomial or some other specific function. The same is true whether testing is replicate or random. We have already explored the use of the pooling concept to derive statistical information which can then be applied with greater levels of confidence to the smaller test data sets.

In view of the conclusion of § 4 *b* (iii), in what follows we use the median as the working life parameter, but we assume that the same principles will apply to any other statistically derived parameter.

Task 2: initial data analysis. For the plotting of constant-life diagrams, a suitable method of interpolation of data such as those in figure 2 and figure 3 is required. There are several possibilities, including linear interpolation between the median data points and nonlinear interpolation, either between the median data points or along a curve fitted to the whole data set for a given R value. Hitherto, we have

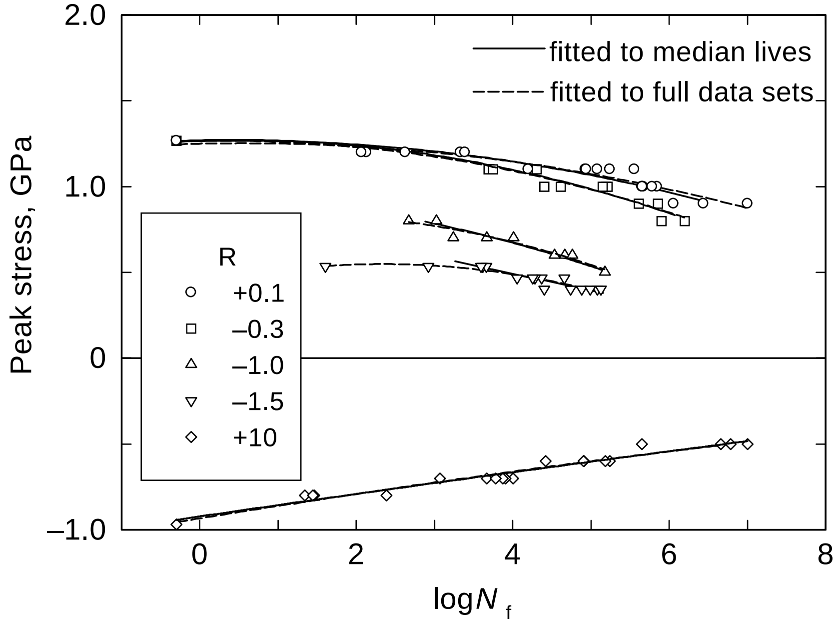

Figure 13. $S/\log N$ data for HTA/913 laminate. The plotted points are the full fatigue data sets and the curves are second-order polynomial fits to the data.

used only nonlinear fitting of polynomials, usually of third order, but it is not yet certain that this is the most appropriate method. We have observed, however, that for a reasonably well-arranged set of data, polynomial fits to the median points and the full data sets are indistinguishable, as can be seen for the data for HTA/913 in figure 13. One advantage of fitting a curve to the full data set is that the extent to which the fitted curves may be safely extrapolated is somewhat greater than when only the median points are used.

One problem that we have observed in fitting polynomials is that when there is a large gap between the monotonic tensile strength and the shortest life for the next highest stress level, the shape of the full curve may not be well represented by the best-fit polynomial. A second-order curve will sometimes rise above the tensile strength value and a third-order curve may become sinuous in the short-life region. In order to avoid this problem, it is tempting to ensure that some fatigue tests are done at higher stress levels. But since tests resulting in lives of between 100 and 1000 cycles are often affected by initial machine stability, a better solution may be to choose a fitting function that allows a fairly linear initial portion. Such models have been suggested by Talreja (1981), Curtis (1986) and Harris *et al.* (1990). Nishijima (1987) has also reviewed a number of similar parametric models. Even linear interpolation would probably be adequate.

Our present method of analysis is to plot graphs of the kind shown in figure 2 in the package Origin, already referred to, which provides a flexible group of curve-fitting routines, including user-defined ones. Once an acceptable level of goodness-of-fit is established, the resulting polynomial coefficients are entered into a spread-sheet (Microsoft Excel), together with values for the monotonic tension and compression strengths of the laminate. Excel then produces a set of data pairs (m, a), as defined by equation (4.12), which includes the end points $(c, 0)$ and $(1, 0)$, representing

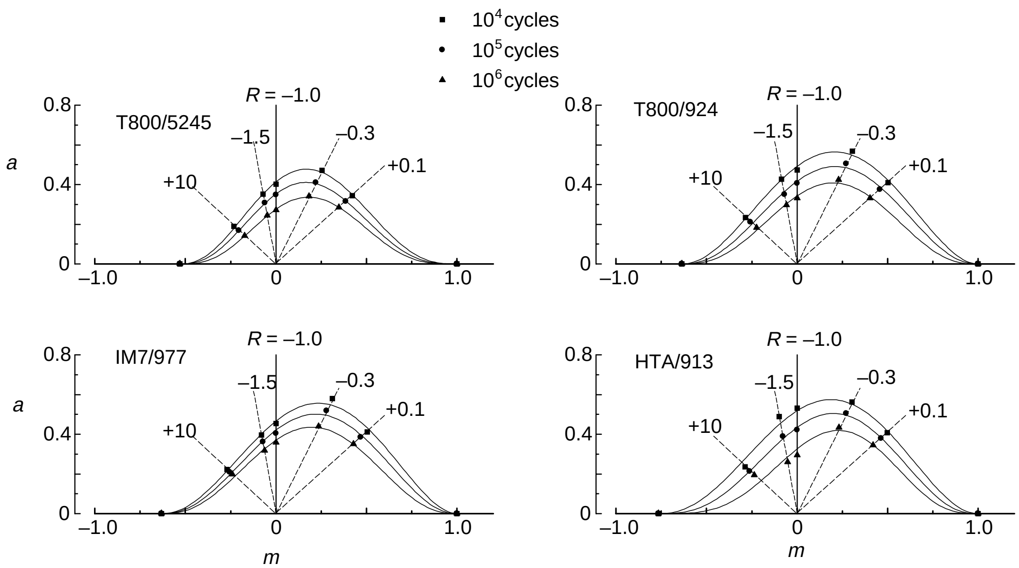

Figure 14. Constant-life plots for four CFRP laminates of $[(\pm 45, 0_2)_2]_S$ construction.

the monotonic failure conditions. These data sets, for given lives (e.g. 10^4, 10^5, 10^6 cycles), are then exported back to Origin for plotting in the form of constant-life diagrams. The extent to which extrapolation of the polynomial curves beyond the actual experimental data window is permissible or safe is a matter which must be carefully controlled. The results of such an analysis for all four experimental laminates are shown in figure 14, from which it can be seen that there is a great deal of similarity between these four materials, despite important differences in actual material characteristics.

Task 3: life prediction. In order to use the constant-life model of equation (4.12) for life prediction, it is first necessary to explore the variation of the fitting parameters with fatigue life. This is done by fitting equation (4.12) to the data sets. In a preliminary fitting session, all three parameters, f, u and v, are allowed to vary freely. The parameter f controls the overall height of the bell-shaped curve, whereas u and v determine the shapes of the left and right wings of the curve and therefore allow for any asymmetry in the material's fatigue response. Early experience suggested that the same value of f could be used for all lives and the mean value from the initial fitting is therefore used in a second fitting session to determine the final values of u and v. Taking the IM7/977 laminate as an example, figure 15 shows the relatively insignificant effect of this assumption on the shapes of the fitted curves. For the fitting processes, the results of which are plotted in figure 14, the coefficients of variation for the fitting of u and v were always of the order of 4%.

As we have remarked, the constant-life plots in figure 14 represent relationships between the parameters a and m, which are already normalized with respect to the laminate tensile strength. It appears, nonetheless, from the results that we have obtained so far, that the scaling parameter, f, is related to the laminate tensile strength, as illustrated in figure 16. The relationship between f and laminate strength is represented by the equation

$$f = 2.78\sigma_f - 2.60, \tag{4.13}$$

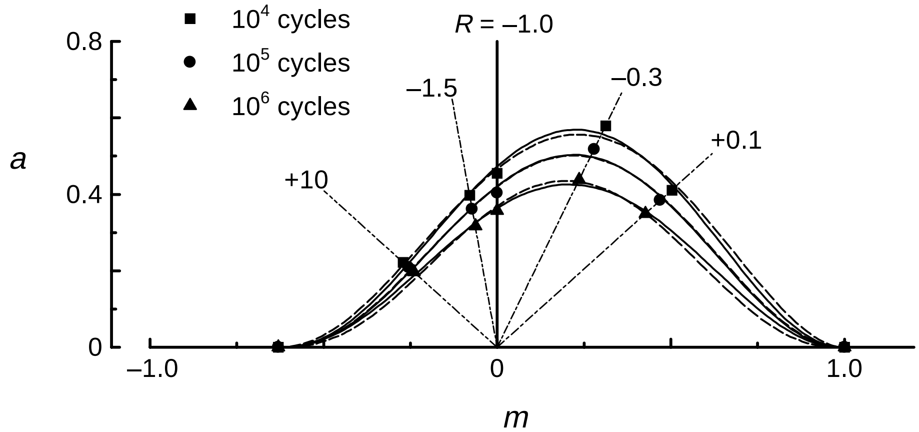

Figure 15. Constant-life plots for IM7/977 composite. The curves represent the relationship $a = f(1-m)^u(c+m)^v$. For the full curves, the three parameters are fitted without constraint, while for the dotted curves, the value of the parameter f has been fixed at the average value of 1.3 obtained by free-fitting.

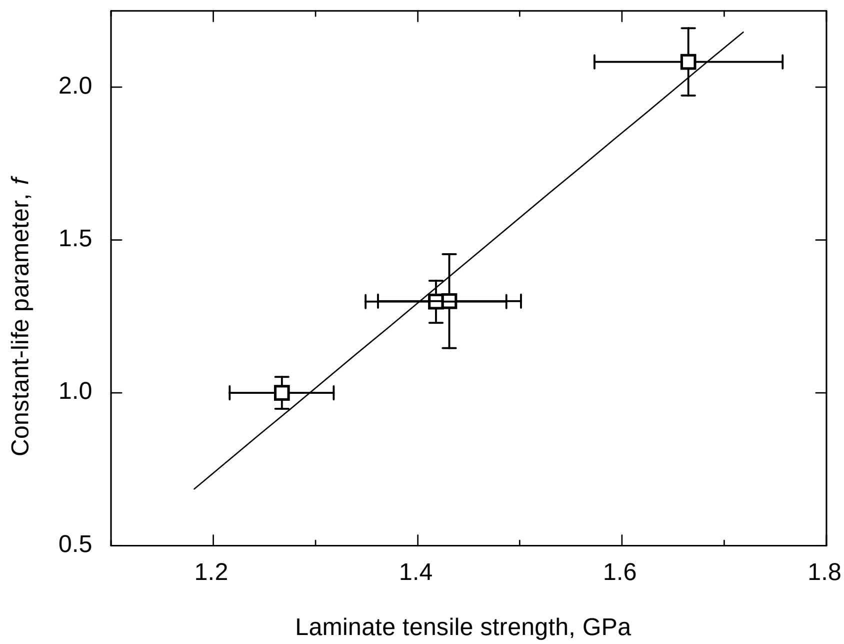

Figure 16. Dependence of constant-life model f parameter on laminate tensile strength. The vertical error bars are standard deviations for the fitted values of f and the horizontal bars are standard deviations for the measured tensile strengths.

with a correlation coefficient of 0.987, which is significant even with only four data pairs. This is unexpected and requires more detailed examination, but the value of a generally valid linear relationship of this kind is that it would provide a useful starting point, based on known information, when the fitting procedure is applied to a new material.

It follows from the derivation of the constant-life plots that the higher the values of f, the better the fatigue performance at any given life, since f, overall, determines the relative 'amplitude' of the curve, and the higher the curve, the greater the alternating stress that can be tolerated for a given mean stress at a given life. This feature therefore reinforces to some extent the common stress–life equal-rank assumption discussed earlier. The regression line in figure 16 does not extrapolate to the origin and it possible, therefore, that the relationship could turn out to be nonlinear if data for a wider range of materials (e.g. other fibres, other lay-ups) were available.

The parameters u and v both depend on the fatigue life, as is apparent from figure 14. The exact nature of this dependence for the four experimental laminates is illustrated in figure 17. The error bars in this figure represent the standard deviations for the fits shown in figure 14. The relationships are again linear over the range of lives studied here and extrapolation to 10^7 cycles would certainly not be too hazardous. For the T800/5245 and IM7/977 laminates, the slopes of the curves of u, v versus $\log N_\mathrm{f}$ are all the same. But whereas the actual u and v values for the IM7/977 are also similar to each other, indicating a symmetric constant-life curve, there is a larger difference between the u and v values for the T800/5245 material than for any of the others. The slopes of the u, v versus $\log N_\mathrm{f}$ lines for T800/924 and IM7/977 are slightly different, the former being somewhat higher, but the actual values for these two laminates are very close indeed, certainly to within the sensitivity of the fit. Thus, in terms of the relationships

$$u, v = A \log N_\mathrm{f} + B, \qquad (4.14)$$

it can be said that the slope, A, is, to all intents and purposes, the same for all three of the higher-performance laminates, while the intercept, B, shows some differences from laminate to laminate and is different for u and v.

The higher the values of u and v, the poorer the fatigue performance, since the further u and v rise above unity (the parabolic special case of the generalized constant-life relationship), the more the 'wings' of the curve are pulled downwards and the more bell-shaped the curve becomes, so reducing the level of alternating stress that can be tolerated for a given mean stress at a given life. It rather appears that the high v values for HTA/913 are associated with the high ratio of the monotonic compression and tensile strengths, defined as the parameter c (equal to $\sigma_\mathrm{c}/\sigma_\mathrm{t}$) since, as already noted, the HTA/913 laminate stands out from the others in this respect.

Finally, the greater the slopes, $\mathrm{d}u/\mathrm{d}\log N_\mathrm{f}$ and $\mathrm{d}v/\mathrm{d}\log N_\mathrm{f}$, the poorer the fatigue performance because the higher the slope, the greater the downward deviation of the $\sigma/\log N_\mathrm{f}$ curve at long lives. As we have already suggested, the greater the difference in the values of u and v for a particular material, the greater the degree of asymmetry of the constant-life curve. This may influence the choice of material if it is known that for a given application a particular degree of compression or tension loading will predominate.

Having established the parameters of the relationships in equations (4.3), it is now possible to predict $\sigma/\log N_\mathrm{f}$ curves for any desired R ratio. This is done by solving the pair of simultaneous equations

$$a = f(1-m)^{u(N_\mathrm{f})}(c+m)^{v(N_\mathrm{f})}, \quad a = m\left(\frac{1-R}{1+R}\right) \qquad (4.15)$$

The first of these is the constant-life equation, equation (4.12), modified to include information about the life-dependence of the two exponents, $u(N_\mathrm{f})$ and $v(N_\mathrm{f})$, as established from the form of equations (4.14). The second is derived from the conventional definition of the stress ratio. Solution of these two equations is easily carried out in a package like the MathSoft Mathcad programme, which will graph or

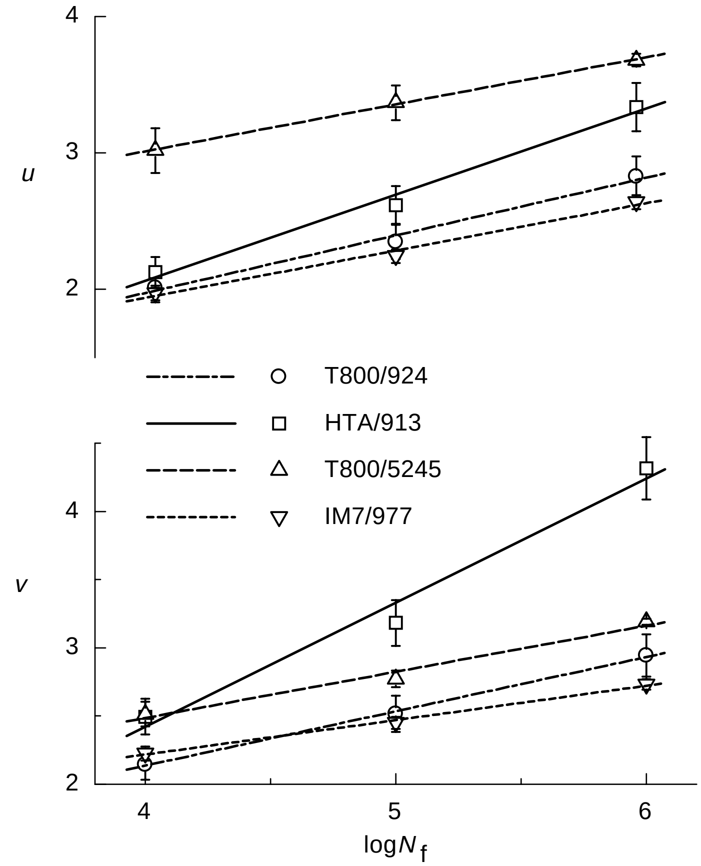

Figure 17. Dependence of the constant-life parameters u and v on loglife for four CFRP laminates.

tabulate $\sigma/\log N_{\mathrm{f}}$ curves for a chosen range of R values. The output can be in one of two forms, depending on the requirements of the user. The first is a three-dimensional generalization of the constant-life plots of figure 11, a surface plot showing the full variation of $(a, m, \log N_{\mathrm{f}})$, as illustrated for the IM7/977 laminate in figure 18. It is interesting to recall that this type of three-dimensional constant-life plot was first used for metallic materials by Stussi in 1955.

Alternatively, a family of stress–life curves of conventional form can be produced for ranges of lives that are consistent with the original experimental data window. As an example of the predictive capability of the method, we reproduce in figure 19 a set of stress–median-life curves for the IM7/977 composite which were predicted at a time when only the monotonic strengths (σ_{t} and σ_{c}) and part of the $\sigma/\log N_{\mathrm{f}}$ curve (the first five data points only) for $R = 0.1$ were available. Choice of values for f, u and v was made from knowledge of the behaviour of the T800/924 laminate already tested. Superimposed on the plot are the full data sets that were subsequently obtained for the IM7/977 laminate; the initial five points used in the prediction are shown as filled symbols. Predictions were made with Mathcad for the range of loglife values from 3 to 7 only, but the full $\sigma/\log N_{\mathrm{f}}$ curves for $R = +0.1$, -0.3 and $+10$ shown in the diagram were drawn by extrapolating back to the monotonic strength values, in the normal way.

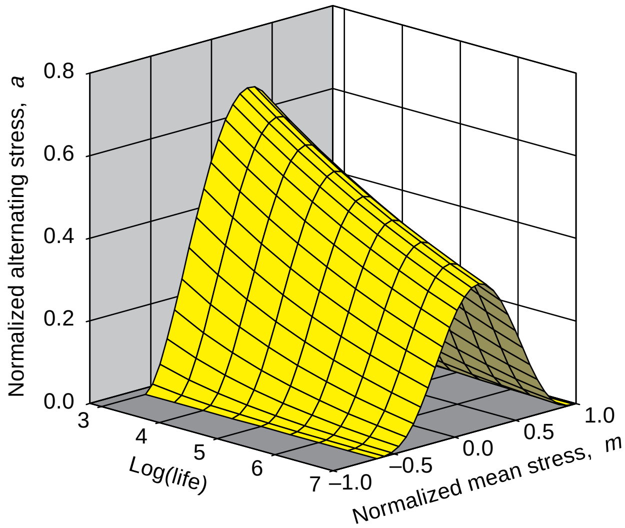

Figure 18. Three-dimensional surface plot of the $a, m, \log N_{\mathrm{f}}$ constant-life relationship defined by the first of equations (4.15) for the $[(\pm 45, 0_2)_2]_{\mathrm{S}}$ IM7/977 laminate.

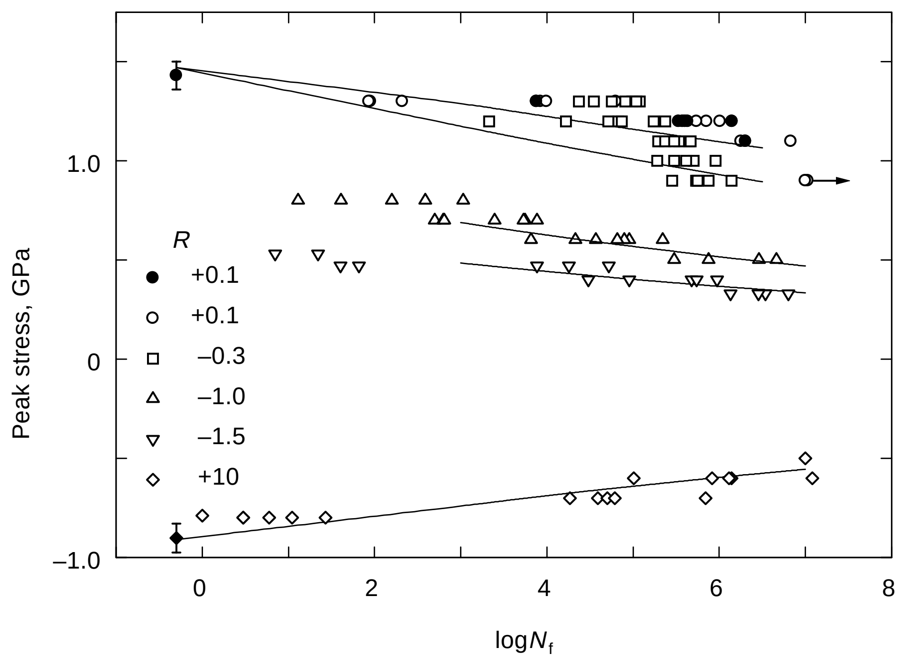

Figure 19. Stress–life curves for the $[(\pm 45, 0_2)_2]_{\mathrm{S}}$ IM7/977 laminate predicted from $\sigma/\log N_{\mathrm{f}}$ data at a single R ratio ($R = 0.1$) and the monotonic strengths. The filled data points for $R = 0.1$ are the values used in the prediction. All other data were obtained after the predictions were made.

It can be seen that that the level of agreement between the predicted curves and the experimental data for stress ratios of -1.0, -1.5 and $+10$ is extremely good. The polynomials for $R = -1.0$ and -1.5 could also have been extrapolated back at least one more decade without danger. The agreement of the predicted curve for $R = +0.1$ with the full data set is acceptable, although the two run-out values at

10^7 cycles have not been allowed for. The poorest fit in figure 19 is for $R = -0.3$. For this laminate, the results at this R value were somewhat different from what we have observed for other composites. The point at which the $\sigma/\log N_f$ curve begins to deviate downwards from the curve for $R = 0.1$ is between 10^4 and 10^5 cycles—perhaps a decade later than is usually the case—and the rate of downward deviation is then quite rapid. As a consequence, the predicted $\sigma/\log N_f$ curve for an R ratio of -0.3 is too conservative over the greater part of the range. From the designer's point of view, however, this is a safe prediction.

(ii) *Prediction of design data*

As a final example, we examine the procedure for predicting a family of minimum-life curves from an existing set of $\sigma/\log N_f$ curves. For this purpose, we take the data for the HTA/913 laminate.

The initial analysis is carried out by means of a simple program written in Basic. The available values of N_f at each stress level, irrespective of R, are first normalized with respect to the actual median life, $m(N_f)$, for that stress level. The normalized data are then pooled, as described earlier, and the Weibull analysis carried out. At this stage, we are still using a two-parameter model, which seems reasonable since the normalized data are effectively bounded at $N_f/m(N_f) = 0$. The regression is carried out twice, for both y on x and x on y, and the geometric mean shape parameter, m, and the value of the gamma function $\Gamma(1 + (1/m))$ are found. The individual life data files are then reprocessed with the established value of m as the appropriate shape parameter and a table of statistical data is printed which includes the mean and median lives, the characteristic and modal values of the probability distribution function for the replicate sample, and the characteristic and modal values for the probability distribution of minima, calculated according to the arguments in § 4 *b* (iii). Data for any required quantile could also equally well be determined.

The minimum-life data (PDF modal values), such as those shown in figure 11, may now be used, with the same procedure as that described in the previous section, to determine a constant-minimum-life diagram. Polynomial functions are fitted to the minimum-life $\sigma/\log N_f$ curves, the coefficients are extracted and these are used, together with the monotonic tensile and compression strengths, to plot the $(a, m, \log N_f)$ Stussi surface. The constant-life function, equation (4.12), is fitted to the constant-life curves to establish the appropriate values of f, u and v, and the life dependences of u and v are obtained. For the HTA/913 data used in this example, the value of f, determined as described earlier, was found to be 1.0, the same value as for the fitting of the median-life results, and the relationships for the life-dependence of u and v were found to be

$$u = 0.65 \log N_f - 0.293, \quad v = 0.82 \log N_f - 0.448. \tag{4.16}$$

Finally, these relationships are used to predict a family of minimum-life $\sigma/\log N_f$ curves as shown in the form of a surface plot in figure 20.

(iii) *Concluding comments*

The life-prediction model discussed here was first conceived as an empirical description of fatigue data obtained over a limited range of R values for some unidirectional composites consisting of members of the hybrid family CFRP/KFRP (Adam *et al.* 1986, 1989; Fernando *et al.* 1988). By inspection, it appeared that a simple parabolic function fitted the normalized (m, a) data for a limited range of R values

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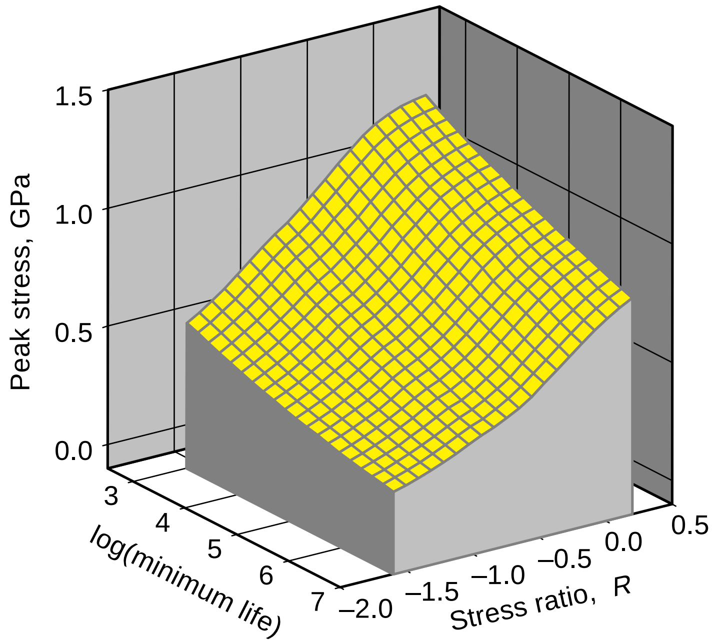

Figure 20. Surface plot of predicted family of stress–life curves for HTA/913 $[(\pm 45, 0_2)_2]_S$ laminate. The lives represented here are minimum-life values derived from the extreme-value model.

and the model was successful in predicting stress–life relationships for a range of compositions within the hybrid family, from plain CFRP to plain KFRP. The potential value of a model which permitted the prediction of fatigue response from a relatively limited experimental data base was emphasized at the time. Since that early work was done, many new high-performance composite materials have been introduced and the obvious requirement was to investigate the validity of the early model for different composite types and different lay-ups.

It became apparent as experiments were carried out over a wider range of R ratios that the original parabolic model was inadequate and it was refined, first, to a symmetric power-law and, subsequently, to an asymmetric power law (Adam *et al.* 1992), as described by equation (4.12). Fatigue results for four modern CFRP laminates (of varying character but all of the same $[(\pm 45, 0_2)_2]_S$ lay-up) are now shown to fit this power-law model very well. The original data for unidirectional hybrids fit the new model acceptably since, as we have shown elsewhere (Gathercole *et al.* 1994), (m, a) data pairs for R ratios between +0.1 and −0.6 fit either model equally well.

A somewhat surprising feature of the results that we have obtained is the apparent similarity of the behaviour of the four composites described in this paper. Only the tensile properties and tension-dominated fatigue response appear to show real differences, which is unexpected, given the known differences in interfacial bond strength in some of these materials, a matter upon which we have also commented elsewhere (Gathercole *et al.* 1994). This similarity means that we have not yet been able to demonstrate unambiguously the generality of the power-law model, but we are now extending the programme to include glass-fibre-reinforced plastics. We have also begun to assess the applicability of the model to materials which have sustained adventitious damage by low-velocity impact.

We also acknowledge the limitations of a model that has been developed on the basis of median fatigue lives and polynomial curve fits and we are at present evaluating the likely validity of the model for use with statistically more realistic parameters. We are interested in offering the designer a conservative predictive tool that will work with a minimum of experimental data but which will allow continuous up-dating of the predictions as more data become available. It seems that a possible approach is to develop a stand-alone expert system. A suggested procedure might be as follows.

An initial experimental programme would be carried out to determine the monotonic tension and compression strengths, together with stress–life data for stress ratios of +0.1 and perhaps −1.2 in order to locate curves reasonably well in both the right- and left-hand quadrants of the constant-life diagram. Perhaps three replicate tests might be done initially at each of three stress levels for both R ratios. By pooling the 15 to 20 fatigue lives so obtained, following the descriptions of Whitney (1981) and of Yang & Jones (1981), a trial value of the Weibull shape parameter, m, could be obtained. A crude estimate could then be made of the modal value of the distribution of minima for each stress level on the basis of the extreme-value-theory premise (Bury 1975; Castillo 1988) that the distribution of the minima of a Weibull distribution with shape parameter m is also a Weibull distribution with shape parameter m and characteristic minimum value $b/n^{1/m}$, where n is the size of the sample and b is the scale parameter of the replicate test set. The minimum-life distribution could then be used to derive trial constant-life curves, with appropriate 'guesses' being made initially for the parameters f (e.g. on the basis of the laminate tensile strength), u and v until sufficient data were available to permit realistic curve fitting.

Initially, the statistical validity of the procedure would be open to question and would require careful testing before serious use but, as further data were acquired, the validity would improve, together with the true predictive capacity of the model. We are currently evaluating the predictive capability of the constant-life model described here against the possible application of an artificial neural network for the same purpose (Lee *et al.* 1995; Harris *et al.* 1996).

5. Conclusions

(1) An investigation has been carried out of the constant-stress fatigue behaviour of a range of modern, high-performance carbon-fibre-composite materials, all with a similar $[(\pm 45, 0_2)_2]_S$ lay-up, the group comprising three different fibres and four different resins.

(2) It is found that, despite the variations in material types and some known differences in the character of the fibre/resin bond in these materials, the fatigue behaviour displayed by the separate materials is remarkably similar, although not such as would permit a simple analysis in terms of the so-called strength/life equal-rank assumption.

(3) The pattern of behaviour observed for these four laminates suggests an approach to the development of a life-prediction model that appears promising. It is based on a constant-life model developed originally for one of the laminates, the T800/5245 material, and it has now been demonstrated that this model fits all of the materials under investigation. Reasonable (and conservative) predictions have been made for one material on the basis of very limited initial fatigue and strength data for that material and accumulated experience obtained with others.

Table 6. *Statistical parameters for fitting with three different ranking methods: full data set for unidirectional HTA/982 at a stress of 1.6 GPa* ($R = 0.1$)

ranking method	slope, m	st. dev. of slope	correlation coeff., r	st. dev. of r
$i/(N+1)$	1.202	0.025	0.9896	0.1705
$(i-0.5)/N$	1.299	0.022	0.9933	0.1472
$(i-0.3)/(N+0.4)$	1.255	0.023	0.9920	0.1561

The authors thank Dr Linda Wolstenholme of City University, UK, for advice on the discussion of the Weibull model. They also thank the Defence Research Agency for support of this work, and for the autoclaving and C-scanning of the laminates, and Mr Matthew Hiley of DRA, Farnborough, for his guidance and encouragement.

Appendix A. Selection of ranking method

A variety of different probability ranking methods have been used for the purpose of Weibull analysis. Three of the more commonly used relationships are

$$\text{the mean rank}\quad F_{\mathrm{W}} = \frac{i}{N+1},$$

$$\text{the median rank}\quad F_{\mathrm{W}} = \frac{i-0.3}{N+0.4},$$

$$\text{probability interval mid-point}\quad F_{\mathrm{W}} = \frac{i-0.5}{N},$$

where i is the rank of a given datum value and N is the number of values in the data set. Discussions of the origins and validity of these and others are to be found in the literature (see, for example, Cunnane 1978; Castillo 1988). We know that most of them are biased to a greater or lesser extent and it is interesting to see how different are the results of their use for estimating the slope of a Weibull plot. The three methods are used to rank the fatigue-life data set for the HTA/982 unidirectional laminate discussed in § 4 *b* (ii) and the results are shown in figure 21 as plots of $\ln[-\ln(1-F_{\mathrm{W}})]$ against $\ln N_{\mathrm{f}}$. The estimated values of the slope, m, and the statistical parameters of the fitting procedure are given in table 6.

Although the probability interval mid-point ranking gives the lowest variances both for the correlation and for the slope, the differences for this data set are very small and the actual values of the slope given by the linear regression differ from each other only in the second decimal place.

In § 4 *b* (ii) we have also examined a data set which represents the minimum values of a set of 20 samples of 7 lives taken at random from the full data set for this material. These data are also represented in figure 8 and it can be seen that they are less well represented by the two-parameter Weibull distribution than the parent data set. When this set of minimum lives is ranked by the three methods described above, the results are as shown in table 7. In this case, the values of the slope differ somewhat more than those for the full data set and the variances for the three different ranking methods are between seven and seventeen times greater. The correlation coefficients

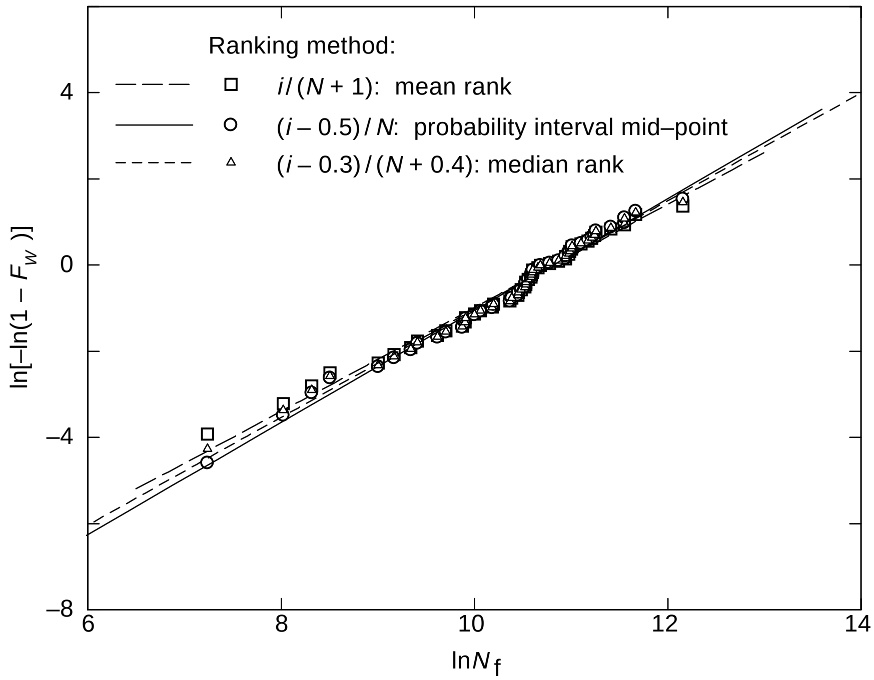

Figure 21. Effect of method of determining plotting position on the estimated value of the Weibull shape parameter. Test data are 50 fatigue life measurements for a ud HTA/982 laminate at a peak stress of 1.6 GPa and $R = 0.1$.

Table 7. *Statistical parameters for fitting with three different ranking methods: minimum-life data*

ranking method	slope, m	st. dev. of slope	correlation coff, r	st. dev. of r
$i/(N+1)$	1.086	0.068	0.9669	0.2858
$(i-0.5)/N$	1.220	0.092	0.9522	0.3905
$(i-0.3)/(N+0.4)$	1.159	0.080	0.9599	0.3376

are still very high, although slightly lower than for the full data set. In this case, it is the mean rank method which gives the lowest variance for the slope.

There appears to be no over-riding case for choosing any one of these ranking relationships and we shall continue using the mean rank method that we have used in the past.

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